The Rigidity Theorems of Hamada and Ohmori, Revisited

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Abstract Let A be a $(0,1)$-matrix of size b by v with $b \geq v$. Suppose that all rows (columns) of A are nonzero and distinct. We show that the rank of A over a field of characteristic 2 satisfies

$$\mathrm{rank}_2(A) \geq \log_2(v+1)$$

with equality if and only if A is the incidence matrix of a point-hyperplane Hadamard design. This generalizes a rigidity theorem of Hamada and Ohmori, who assumed that $v+1$ is a power of 2 and that A is already known to be the incidence matrix of a Hadamard design. Our results follow from a generalization of a rank inequality of Wallis.

1 Introduction and Definitions

In this paper we observe that some inequalities in the literature on the ranks of incidence matrices of Hadamard designs apply to a wider class of matrices and that the proofs are somewhat simpler in the more general context. In particular, we show that an analysis of the case of equality of an extension of a rank inequality of Wallis yields generalizations of two rigidity theorems of Hamada and Ohmori.

Let A be a rectangular matrix with entries in a field F. Let $\mathrm{rank}_F(A)$ denote the rank of A over F. The **p-rank** of A is the rank of A over the field Z_p of integers modulo p and is denoted by

$$\mathrm{rank}_p(A).$$

Let us say that the matrix A is **column-projective** over F provided its columns are nonzero and no column is a multiple of another column. Also, A is **projective** provided both A and its transpose A^T are column-projective. Suppose that A is $(0,1)$-matrix, that is, a matrix with each element in the set $\{0,1\}$. Then A is column-projective if and only if its columns are distinct and nonzero; this property is independent of the field F.

Combinatorial interest in p-ranks stems in part from their use in the study of linear codes associated with incidence matrices of block designs. Let F_q denote a field with q elements, where q is a power of the prime p. Let A

be the incidence matrix of a square balanced incomplete block design with parameters (v, k, λ). Thus A is a $(0,1)$-matrix of size v by v that satisfies

$$A^T A = (k - \lambda)I + \lambda J \qquad \text{and} \qquad AJ = kJ,$$

where J denotes an all 1's matrix of an appropriate size. The codewords of the q-ary linear code $\mathcal{C}_q(A)$ generated by A are the linear combinations over F_q of the row vectors of A. Clearly the length and dimension of $\mathcal{C}_q(A)$ are v and $\text{rank}_p(A)$, respectively. One may show that $\mathcal{C}_q(A)$ corrects $\lfloor k/2\lambda \rfloor$ errors [2]. In this context it is natural to assume that A is (column)-projective.

2 Two Rigidity Theorems

We now state our generalizations of two rigidity theorems of Hamada and Ohmori. The proofs appear in § 4. The point-hyperplane design $\mathbf{D}_s$ in the projective geometry $\text{PG}(s-1, 2)$ and its complement $\overline{\mathbf{D}}_s$ play special roles in these theorems. We recall some basic properties of these designs in § 3.

Theorem 1. *Let A be a projective $(0,1)$-matrix of size b by v with $b \geq v$. Then*

$$\text{rank}_2(A) \geq \log_2(v+1) \tag{1}$$

with equality if and only if $b = v$ and A is an incidence matrix of the complement $\overline{\mathbf{D}}_s$ of the point-hyperplane design, where $s = \log_2(v+1)$.

Remark. Hamada and Ohmori [1] established Theorem 1 under much stronger hypotheses; they assumed that A is the incidence matrix of a square block design with parameters of the form

$$(v, k, \lambda) = (2^s - 1, 2^{s-1}, 2^{s-2}), \tag{2}$$

that is, a **Hadamard design.** Their proofs seemingly depend on deeper results on the ranks of incidence matrices of block designs associated with projective geometries. We remove these apparent dependencies and show that a specific design-theoretic structure is *forced* when equality holds in (1). This characterization of a combinatorial structure from the value of a single parameter is the hallmark of a rigidity theorem.

Theorem 2. *Let A be a $(0,1)$-matrix of size b by v with $b \geq v$. Suppose that $J - A$ is projective, b is odd, and each row and column of A has an odd number of 1's. Then*

$$\text{rank}_2(A) \geq \log_2(v+1) \ + \ 1 \tag{3}$$

with equality if and only if $b = v$ and A is an incidence matrix of the point-hyperplane design $\mathbf{D}_s$, where $s = \log_2(v+1)$.

Remark. Hamada and Ohmori [1] established Theorem 2 under the stronger hypothesis that A is the incidence matrix of the complement of a Hadamard design with parameters of the form (2).

3 A Rank Inequality

Theorem 3. *Let A be a column-projective $(0,1)$-matrix with v columns. Then over any field F*

$$\mathrm{rank}_F(A) \geq \log_2(v+1). \tag{4}$$

Equality holds if and only if A has a column-projective submatrix of size s by $2^s - 1$, where $s = \mathrm{rank}_F(A)$.

Remark. Wallis [3] used an elaborate inductive scheme of row and column operations to prove an inequality equivalent to (4) under much stronger hypotheses; he assumed that A is a $(0,1)$-matrix of size v by v that is nonsingular over the field of rationals. (Also see his book [4], pp 168-170.) Moreover, he did not characterize equality. Our direct proof is based on a counting argument and leads to the stated characterization of equality, which in turn leads to proofs of Theorems 1 and 2.

Proof. Let $s = \mathrm{rank}_F(A)$. Without loss of generality the leading s by s submatrix of A has rank s. Then two columns of A are distinct and nonzero if and only if they are distinct and nonzero in their leading s positions. There are exactly $2^s - 1$ nonzero column vectors of 0's and 1's with s components. Thus $v \leq 2^s - 1$. This proves (4) with the stated characterization of equality. □

4 The Extremal Designs

In this section we recall properties of the extremal designs that arise in Theorems 1 and 2. Let $\overline{\mathbf{D}}_s$ denote the complement of the point-hyperplane design in the projective geometry $\mathrm{PG}(s-1,2)$. Let $\overline{A}_s$ be the incidence matrix of $\overline{\mathbf{D}}_s$. The columns of $\overline{A}_s$ correspond to the $2^s - 1$ nonzero vectors (points) in an s-dimensional vector space over Z_2, while the rows correspond to the $2^s - 1$ complements of the $(s-1)$-dimensional subspaces (blocks). Containment defines incidence in $\overline{D}_s$. Without loss of generality

$$\overline{A}_s = \left[\begin{array}{c|c} N_s & M \\ \hline * & * \end{array}\right],$$

where the leading s by s submatrix N_s is nonsingular. The columns of the submatrix $[N_s|M]$ consist of all $2^s - 1$ nonzero linear combinations of the columns of N_s. Now the symmetric difference of two blocks of $\overline{\mathbf{D}}_s$ is also a block; this is a defining property of the design $\overline{\mathbf{D}}_s$. It follows that the rows of $\overline{A}_s$ are the nonzero linear combinations of the rows of $[N_s|M]$. One may verify that $\overline{A}_s$ is a $(0,1)$-matrix of size $2^s - 1$ by $2^s - 1$ that satisfies

$$A^T A = 2^{s-2}(I + J) \qquad \text{and} \qquad AJ = 2^{s-1}J,$$

and thus that $\overline{\mathbf{D}}_s$ is indeed a square block design with parameters

$$(v, k, \lambda) = (2^s - 1, 2^{s-1}, 2^{s-2}).$$

Clearly the incidence matrix $\overline{A}_s$ is determined by the parameter s up to row and column permutations. Also,

$$\mathrm{rank}_2(\overline{A}_s) = s.$$

The complementary design $\mathbf{D}_s$ has incidence matrix $J - \overline{A}_s$ and parameters

$$(v, k, \lambda) = (2^s - 1, 2^{s-1} - 1, 2^{s-2} - 1).$$

Both $\overline{\mathbf{D}}_s$ and $\mathbf{D}_s$ satisfy $v = 4(k - \lambda) - 1$ and hence are Hadamard designs.

5 Proofs of Theorems 1 and 2

Proof of Theorem 1. Apply Theorem 3 with $F = Z_2$ to deduce that $\mathrm{rank}_2(A) \geq \log_2(v + 1)$. Suppose that equality holds, say, $v = 2^s - 1$, where $s = \mathrm{rank}_2(A)$. Then A has a column-projective submatrix of size s by $2^s - 1$. The projectivity of A and the inequality $b \geq v$ imply that $b = v = 2^s - 1$, and thus the rows and columns of A are determined up to permutations, as in the discussion of the matrix A_s in § 4; the characterization of equality follows. □

Theorem 2 follows immediately from Theorem 1 and the following lemma.

Lemma. *Let A be a $(0, 1)$-matrix of size b by v. Suppose that b is odd and that each row and column of A has an odd number of 1's. Then*

$$\mathrm{rank}_2(A) = \mathrm{rank}_2(J - A) \ + \ 1.$$

Proof. The hypotheses imply that we may transform A as follows without altering its 2-rank: Append a column of 1's, and then append a row of 1's to the resulting matrix to obtain a bordering of A of size $b+1$ by $v+1$. Now add column $v + 1$ to each of the first v columns, and then subtract row $b + 1$ from each of the first b rows. The resulting matrix is the direct sum $(J + A) \oplus [1]$, which clearly has 2-rank equal to $\mathrm{rank}_2(J - A) + 1$. □

6 The Smith Normal Form

Let A be an integral matrix of size b by v and rank r over the field of rationals. Then A may be transformed by elementary row and column operations to a diagonal matrix

$$S_A = \mathrm{diag}[a_1, a_2, \ldots, a_r, 0, \ldots, 0],$$

known as the **Smith normal form** of A, with the property that a_i divides a_{i+1} for $i = 1, \ldots, r-1$. The diagonal elements $a_1, \ldots, a_r, 0, \ldots, 0$ are the **invariant factors** of A and are uniquely determined up to sign. The p-rank of A is related to the invariant factors $a_1, \ldots, a_r$ in a simple manner:

$$\text{rank}_p(A) = \max\{i : \ p \text{ does not divide } a_i\}. \tag{5}$$

Our generalization in Theorem 3 of a result of Wallis leads directly to the following two theorems, which extend his work in [3], [4]. The first is an immediate consequence of (5) and Theorem 3.

Theorem 4. *The invariant factors of a column-projective* $(0,1)$*-matrix* A *with* v *columns satisfy* $a_1 = \cdots = a_s = 1$ *for some* $s \geq \log_2(v+1)$.

Theorem 5. *Let* H *be a* $(1,-1)$*-matrix with* v *columns, none of which is a multiple of any other. Then for some* $s \geq \log_2(v)$ *the Smith normal form of* H *is of the form*

$$S_H = \text{diag}[1, \overbrace{2, \ldots, 2}^{s}, *, \ldots, *].$$

Proof. We may multiply suitable rows and columns of H by -1 so that all elements in the first row and column are 1. Now subtract column 1 from all other columns, and then subtract row 1 from all other rows to transform H to a matrix $[1] \oplus (2A)$, where A is a $(0,1)$-matrix with $v-1$ columns. The hypothesis on the columns of H implies that A is a column-projective, and the result follows from Theorem 4. □

References

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