Differential Equations - SM212
Final Exam Review Notes

1. Classification

   - Order
   - Linearity: $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = f(x)$, where dependent variable y and its derivatives have no nonlinear operations (e.g., squaring) performed on them,
   - Separable
   - Autonomous

2. Graphical approximations to the solution to a first order ODE

   - Direction fields and isoclines
   - Autonomous DEs
     - $y' = f(y)$,
     - Find critical points and sketch phase portrait
     - Types of equilibria
       (a) stable - attractor
       (b) unstable - repellor

3. Numerical methods for $y' = f(x, y)$

   - Euler's method
     -
       $$y_{new} = y_{old} + hf(x_{old}, y_{old}), \quad x_{new} = x_{old} + h$$
       or
       $$y_{n+1} = y_n + hf(x_n, y_n), \quad x_{n+1} = x_n + h$$
     - successive "tangent line"/linear approximation using slope = $y' = f(x, y)$
   - Improved Euler's method
     - If $y^*_{new} = y_{old} + hf(x_{old}, y_{old})$ then
       $$y_{new} = y_{old} + \frac{h}{2}[f(x_{old}, y_{old}) + f(x_{new}, y^*_{new})], \quad x_{new} = x_{old} + h$$

- successive *averaged* "tangent line"/linear approximation using slope $= y' = f(x,y)$

4. First Order Methods of Solution

   - Separation of Variables $y' = f(x)/g(y) \implies \int g(y)\,dy = \int f(x)\,dx$
   - Integrating Factor
     - $y' + p(x)y = q(x)$
     - $\mu = e^{\int p(x)\,dx}$
     - $\mu y' + p(x)\mu y = (\mu y)' = \mu f(x)$
     -

$$y = \frac{\int \mu f(x)\,dx + C}{\mu}$$

5. Applications

   - Exponential Growth/Decay
     - General $y' = Ay$, $y(0) = y_0 \implies y = y_0 e^{At}$
     - Radioactive Decay $m(t) = m_0(\frac{1}{2})^{t/t_{1/2}}$
   - Heating/Cooling
     $T' = k \cdot (T - T_{room})$, where $T = T(t)$ is the temperature of the object and T_{room} (which can depend on t) is temperature of the room (or environment or medium)
   - Mixing $A' = F_{in}C_{in} - F_{out}\frac{A}{\text{Tank}(t)}$, where $F_{in/out}$ is the flow rate of the solution flowing in/out, C_{in} is the concentration of the solution pouring in, Tank(t) is the volume of solution in the tank and time t, and $A = A(t)$ is the amount (mass) of solute.
   - Falling Body $mv' + kv = mg$, where $k \geq 0$ is the coefficient of air resistance.
   - Circuits
     - RL $L\frac{di}{dt} + Ri = e(t)$, solve for $i = i(t)$
     - RC $R\frac{dq}{dt} + \frac{1}{C}q = e(t)$, solve for $q = q(t)$

6. Higher Order DE Underlying Theory

- n^{th} Order Differential Equation
  Initial Value Problem, subject to n initial conditions

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = f(x),$$

  Initial Conditions: $y(x_0) = y_0$, $y'(x_0) = y_1$, $\ldots$, $y^{(n-1)}(x_0) = y_{n-1}$.

- n^{th} Order Homogeneous Differential Equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = 0.$$

  - There are exactly n *fundamental solutions*, $y)1$, y_2, $\ldots y_n$
  - Fundamental solutions are linearly independent
  - General solution: $y = c_1y_1 + \ldots c_ny_n$, for arbitrary constants (sometimes called "parameters") c_1, ..., c_n

- n^{th} order non-homogeneous differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = f(x),$$

  General solution: $y = y_h + y_p$, where

  - y_h is a solution to

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = 0,$$

  - y_p is *any* solution to

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = f(x).$$

7. Methods of Solution for Second Order Linear Differential Equations

- Homogeneous with Constant Coefficients
  - Factor characteristic polynomial (sometimes called the *auxiliary equation*)
  - Roots, r_i , yield members of Fundamental Set $y_k = e^{r_kx}$
  - For roots repeated k times, $y_1 = e^{rx}$, $y_2 = xe^{rx}$, $y_3 = x^2e^{rx}$, $\ldots$, $y_k = x^{k-1}e^{rx}$,

  - For complex conjugate roots, $r = \alpha + i\beta$, $y_1 = e^{\alpha x}\cos(\beta x)$, $y_2 = e^{\alpha x}\sin(\beta x)$.

- Non-homogeneous with Constant Coefficients (undetermined coefficients and annihilators)

  These methods only applies to the non-homogeneous constant coefficient ODEs

  $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = f(x)$$

  where the "forcing function" $f(x)$ is "elementary". More precisely, $f(x)$ must be a sum of terms which are a product of polynomials, exponentials, sin's and/or cos's.

  Undetermined coefficients:

  - First find y_h, the solution to the homogeneous equation
  - Next find the repeated derivatives of the forcing $f(x)$, writing down all the individual terms separately, removing constant factors which might be multiplying such terms. By the hypothesis on $f(x)$, only a finite number of such functions can arise.
  - "Guess" for y_p a linear combination of such functions. The coefficients in this linear combination are the "undetermined coefficients". Multiply by x those terms which "agree" with any terms in y_h.
  - Plug y_p into the ODE and solve for the undetermined coefficients
  - $y = y_h + y_p$
  - Solve for the "parameters" $c_1, \ldots, c_n$ if you are given ICs.

  Annihilator method: Write

  $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = f(x)$$

  symbolically as $L(y) = f(x)$.

  - Find the Annihilator of $f(x)$ - the differential operator L_0 of smallest degree such that $L_0(f(x)) = 0$.
  - Multiply annihilators for sums of functions
  - Find y_h - the solution to $L(y) = 0$.

- Find solutions of $L_0(L(y)) = 0$.
    - identify terms which comprise y_h
    - remaining terms comprise y_p
    - $y = y_h + y_p$
- Use $L(y) = f(x)$ to solve for coefficients in y_p
- Solve for "parameters" of y_h using initial conditions, if given.

8. Applications

- Free Undamped Motion
  $mx'' + kx = 0$
  if $\omega = \sqrt{k/m}$ then

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \sin(\omega t + \phi)$$

  - ω =angular speed
  - $P = 2\pi/\omega$ = period
  - $f = 1/P$= frequency
  - $\phi = 2\tan^{-1}(\frac{c_1}{c_2+A})$ = phase angle
  - $A = \sqrt{c_1^2 + c_2^2}$ = amplitude
- Free Damped Motion
  $mx'' + bx' + kx = 0$
  Roots r_1, r_2 of $mD^2 + bD + k = 0$:
  - real, distinct $\Longrightarrow$ over-damped, $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$,
  - repeated root $r = r_1 = r_2 \Longrightarrow$ critically damped, $x = c_1 e^{rt} + c_2 t e^{rt}$,
  - complex conjugate roots $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta \Longrightarrow$ under-damped $x = c_1 e^{\alpha t} \cos(\omega t) + c_2 e^{\alpha t} \sin(\omega t)$
- Forced Motion
  $mx'' + bx' + kx = f(t)$, where $f(t)$ is the external force acting on the spring-mass system.
  Roots r_1, r_2 of $mD^2 + bD + k = 0$
  - Solve as other non-homogeneous equations: write solution as $x = x_h + x_p$, where

    - x_h is *transient term*
    - x_p is *steady state term*
  - Undamped forced motion will resonate if "forced natural frequency" equals "natural frequency". In other words, if $f(t) = f_0 \sin(\gamma t)$ or $f(t) = f_0 \cos(\gamma t)$, for some γ, the *resonance* occurs if and only if $\gamma = \sqrt{k/m}$ (assuming $b = 0$).

- RLC Electric Circuit

$$Lq'' + Rq' + \frac{1}{C}q = e(t)$$

  where $e(t)$ is the battery or EMF.

  - Solve using same method as Forced Damped Motion
  - Dont forget that $i = q'$
  - If you write solution as $q = q_h + q_p$, where
    - q_h is the *transient charge*
    - q_p is the *steady state charge*

9. Laplace transforms

- The Laplace Transform converts a constant-coefficient linear differential equation in the independent variable t to an algebraic equation in independent variable s.
- The Laplace Transform is an integral transform.

  Know the definition and be familiar with the Laplace Transforms of the common functions (polynomials, sines, cosines, exponentials).
- Properties of the Laplace Transform:
  - Translation theorem 1: $\mathcal{L}[e^{at}f(t)](s) = F(s-a)$
  - Translation theorem 2: $\mathcal{L}[f(t-a)u(t-a)](s) = e^{-as}F(s)$
    Think of $u(t-a)$ as a mathematical switch, which turns ON at $t = a$
  - Derivative theorem 1: $\mathcal{L}[f'(t)](s) = sF(s) - f(0)$, and similar formulas for $f''(t)$, $f'''(t)$, ...
  - Derivative theorem 2: $\mathcal{L}[tf(t)](s) = -F'(s)$, and similar formulas for $t^2f(t)$, $t^3f(t)$, ...

- DO NOT get confused and take the products of the Laplace Transforms!
- Use Laplace transforms to solve initial value problems
  - Take Laplace Transform of entire problem
  - . Solve for Laplace Transform of dependent variable
  - Take Inverse Laplace Transform for solution to IVP
- Convolution theorem (the Laplace transform of the convolution is the product of the Laplace transforms)
  - Know the definition of the convolution
  - Can use the convolution theorem to solve second order linear ODEs with constant coefficients

$$ay' + by' + cy = f(t), \quad y(0) = y'(0) = 0,$$

$$y(t) = (h * f)(t),$$

where $f(t)$ is the forcing function and

$$h(t) = \mathcal{L}^{-1}[\frac{1}{as^2 + bs + c}](t)$$

is called the *impulse response function.*

10. Matrix operations

- Basics
  - Augmented Matrix represents a system of equations. For example, the 2×2 linear system in standard form (with all the unknowns on the left hand side and the remaining known quantities on the right hand side)

$$\begin{cases} ax + by = r_1 \\ cx + dy = r_2 \end{cases}$$

can be represented as its "2×3 matrix of coefficients"

$$A = \begin{pmatrix} a & b & r_1 \\ c & d & r_2 \end{pmatrix}$$

- In *row reduced echelon form* (rref), the above is reduced to a much "sparser" matrix, rref(A), which represents the matrix of coefficients of a much simpler linear system.
- Eigenvalues and eigenvectors. The *eigenvalue equation* forms the definition: $A\vec{v} = \lambda\vec{v}$. This means $\vec{v}$ is an eigenvector of A with eigenvalue λ. An $n \times n$ matrix has n eigenvalues, counted according to multiplicity: they are the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.

- Application of rref to systems of linear ordinary differential equations with constant coefficients.
  - Take the Laplace transform of all the equations and put it in standard form.
  - Compute the row reduced echelon form of its augmented matrix.
  - Solve for the Laplace transforms of the dependent variables.
  - Take inverse Laplace transforms to solve the system of ODEs.
- Application of eigenvalues and eigenvectors to systems of linear ordinary differential equations with constant coefficients.
  - Put the systems in matrix form: $\vec{X}' = A\vec{X}$, where $\vec{X} = \vec{X}(t)$ is the vector of n unknown functions (the dependent variables of the system) and A is an $n \times n$ matrix of constants.
  - Compute the eigenvalues λ_1, ..., λ_n, and their corresponding eigenvectors $\vec{v}_1$, ..., $\vec{v}_n$.
  - If all the eigenvalues are distinct, the solution is

$$\vec{X} = c_1\vec{v}_1 e^{\lambda_1 t} + ... + c_n\vec{v}_n e^{\lambda_n t},$$

    for arbitrary constants (or "parameters") c_1, ..., c_n.
  - If there are initial conditions, solve for the c_1, ..., c_n.
- Applications
  - Electrical Networks.
    Determine System of independent ODEs using Kirchoffs Laws for current Loops and nodes.
  - Lanchester's equations

If the X-men are battling the Y-men in a simple conventional battle then

$$\begin{cases} x' = -Ay \\ y' = -Bx \end{cases}$$

can be used to model the number $x = x(t)$ of X-men at time t and the number $y = y(t)$ of Y-men at time t.

- Numerical methods
  - Eulers method for systems

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2), \quad y_1(a) = c_1, \\ y_2' &= f_2(x, y_1, y_2), \quad y_2(a) = c_2. \end{aligned}$$

Use

$$y_{1,new} = y_{1,old} + hf_1(x_{1,old}, y_{1,old}, y_{2,old}),$$

$$y_{2,new} = y_{2,old} + hf_2(x_{1,old}, y_{1,old}, y_{2,old}),$$

and $x_{new} = x_{old} + h$.
For a 2nd order ODE,

$$y'' + p(x)y' + q(x)y = f(x), \qquad y(a) = c_1, \quad y'(a) = c_2,$$

do the following:

* write 2nd Order ODE as a system of two 1st order DEs. Using $y_1 = y$ and $y_2 = y'$,

$$\begin{aligned} y_1' &= y_2, & y_1(a) = c_1, \\ y_2' &= f(x) - q(x)y_1 - p(x)y_2, & y_2(a) = c_2. \end{aligned}$$

* Apply Euler's method for systems as above.

11. Fourier Series

Represents "any" function on an interval $(-L, L)$ centered at the origin as a convergent series of orthogonal functions.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\ dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\ dx.$$

12. Half-Range Fourier Series

Represents a function defined on an interval $(0, L)$.

- Fourier Cosine Series

  Also used as Fourier series for EVEN functions.

  To have a cosine series you must be given two things: (1) a "period" $P = 2L$, (2) a function $f(x)$ defined on the interval of length L, $0 < x < L$.

  $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}),$$

  where

  $$a_n = \frac{2}{L}\int_0^L \cos(\frac{n\pi x}{L}) f(x)\ dx.$$

- Fourier Sine Series

  Also used as Fourier series for ODD functions

  To have a sine series you must be given two things: (1) a "period" $P = 2L$, (2) a function $f(x)$ defined on the interval of length L, $0 < x < L$.

  $$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}),$$

  where

  $$b_n = \frac{2}{L}\int_0^L \sin(\frac{n\pi x}{L}) f(x)\ dx.$$

13. Separation of variables for 1st and 2nd order linear homogeneous PDEs

$$a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4 U_x + a_5 U_y + a_6 u = 0, \qquad u = u(x, y),$$

where the coefficients a_i could depend on x or y.

Typical examples:

- advection equation

$$u_x + c u_t = 0$$

- heat equation

$$k u_{xx} = u_t$$

- wave equation

$$\alpha^2 u_{xx} = u_{tt}$$

14. Heat equation

$$\begin{cases} k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \\ u(x,0) = f(x). \end{cases}$$

Here $u(x,t)$ denotes the temperature at a point x on the wire at time t, so $f(x)$ is the wire's initial temperature.

- zero ends

$$\begin{cases} k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \\ u(x,0) = f(x), \\ u(0,t) = u(L,t) = 0. \end{cases}$$

  - Find the sine series of $f(x)$:

$$f(x) \sim \sum_{n=1}^{\infty} b_n(f) \sin(\frac{n\pi x}{L}),$$

  - The solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(f) \sin(\frac{n\pi x}{L}) \exp(-k(\frac{n\pi}{L})^2 t).$$

- insulated ends

$$\begin{cases} k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \\ u(x,0) = f(x), \\ u_z(0,t) = u_x(L,t) = 0. \end{cases}$$

- Find the cosine series of $f(x)$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(\frac{n\pi x}{L}),$$

- The solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(\frac{n\pi x}{L}))\exp(-k(\frac{n\pi}{L})^2 t).$$

15. Wave equation

    The wave equation with zero ends boundary conditions models the motion of a (perfectly elastic) guitar string of length L:

$$\begin{cases} \frac{\partial^2 w(x,t)}{\partial x^2} = a^2\frac{\partial^2 w(x,t)}{\partial t^2} \\ w(0,t) = w(L,t) = 0. \end{cases}$$

    Here $w(x,t)$ denotes the displacement from rest of a point x on the string at time t. The initial displacement $f(x)$ and initial velocity $g(x)$ at specified by the equations

$$w(x,0) = f(x), \qquad w_t(x,0) = g(x).$$

- Find the sine series of $f(x)$ and $g(x)$:

$$f(x) \sim \sum_{n=1}^{\infty} b_n(f)\sin(\frac{n\pi x}{L}), \qquad g(x) \sim \sum_{n=1}^{\infty} b_n(g)\sin(\frac{n\pi x}{L}).$$

- The solution is

$$w(x,t) = \sum_{n=1}^{\infty}(b_n(f)\cos(\frac{n\pi t}{aL}) + \frac{aLb_n(g)}{n\pi}\sin(\frac{n\pi t}{aL}))\sin(\frac{n\pi x}{L}).$$