Partial fractions[1]

Partial fractions is a method of rewriting a rational function[2] as a sum of simpler rational functions. The usual applications are to computing integrals and inverse Laplace transforms of rational functions.

To apply the PFDs to a quotient of polynomials $N(x)/D(x)$, the degree of the numerator $N(x)$ must be strictly smaller than the degree of the denominator $D(x)$. If this is not the case, you must first use long division to reduce the degree of the numerator. Let $N(x)$ be a polynomial of lower degree than another polynomial $D(x)$.

The *partial fraction decomposition* (PFD) of $N(x)/D(x)$:

(1) Factor $D(x)$ into irreducible factors having real coefficients. Now $D(x)$ is a product of distinct terms of the form $(ax+b)^r$ or irreducible $(ax^2+bx+c)^s$, for some integers $r > 0$, $s > 0$. For each term $(ax+b)^r$ the PFD of $N(x)/D(x)$ contains a sum of terms of the form

$$\frac{A_1}{(ax+b)} + \cdots + \frac{A_r}{(ax+b)^r}$$

for some constants A_i, and for each term $(ax^2+bx+c)^s$ the PFD of $N(x)/D(x)$ contains a sum of terms of the form

$$\frac{B_1 x + C_1}{(ax^2+bx+c)} + \cdots + \frac{B_s x + C_s}{(ax^2+bx+c)^s},$$

for some constants B_i, C_j. $\frac{N(x)}{D(x)}$ is the sum of all these simpler rational functions.

(2) Now you have an expression for $\frac{N(x)}{D(x)}$ which is a sum of simpler rational functions. The next step is to solve for the coefficients A's, B's, C's occurring in the numerators. Cross multiply both sides by $D(x)$ and expand out the resulting polynomial identity for $N(x)$ in terms of the coefficients A's, B's, C's. Equating coefficients of powers of x on both sides gives rise to a linear system of equations for the A's, B's, C's which you can solve.

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[2]A rational function is a quotient of polynomials. For example, $\frac{x^2+1}{x^3-3x+2}$ is a rational function but $\frac{x-3}{x^{1/2}-2+x}$ is not.

Problem: Find the partial fractions decomposition for $\frac{5x-2}{2x-x^2}$.

Solution: We factor the denominator, set up the partial fractions and solve for the coefficients.

$$\frac{5x-2}{2x-x^2} = -\frac{5x-2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}.$$

Multiply through by the denominator and plug in values for x or use the cover up method to solve for A and B:

$$-(5x-2) = A(x-2) + Bx.$$

Plugging in $x = 0$ we get $A = -1$ and plugging in $x = 2$ we get $B = -4$. Thus,

$$\frac{5x-2}{2x-x^2} = \frac{-1}{x} + \frac{-4}{x-2}.$$

Problem[3]: Find the general form of the PFD for

$$\frac{2x^6 - 4x^5 + 5x^4 - 3x^3 + x^2 + 3x}{(x-1)^3(x^2+1)^2}.$$

Solution: The partial fraction decomposition takes the form

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}.$$

Problem: Find the PFD of

$$\frac{25}{x^2(5-x)}.$$

Solution: The general form of the PDF in this case is

$$\frac{25}{x^2(5-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(5-x)}$$

Multiplying through by the denominator gives: $25 = Ax(5-x) + B(5-x) + Cx^2$. Plugging in $x = 0$ and $x = 5$ we get B and C for free: $B = 5$ and $C = 1$. This reduces the equation to $25 = Ax(5-x) + 5(5-x) + x^2$. Plugging in $x = 1$, we have $25 = 4A + 20 + 1$, so that $A = 1$ as well.

[3] This example is completely worked out in the Wikipedia article on "Partial fraction decomposition."

Exercise 1: The general form of the partial fraction decomposition for

$$\frac{s+1}{(s^4-1)}.$$

is

(a) $\frac{As^3+Bs^2+Cs+D}{s^4-1}$, (b) $\frac{As+B}{s^2+1}+\frac{C}{s+1}+\frac{D}{s-1}$, (c) $\frac{A}{s^2+1}+\frac{B}{s+1}+\frac{C}{s-1}$,
(d) $\frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+2}+\frac{D}{s-2}$, (e) none of the above.

Exercise 2: Solve for A, B, C in the partial fraction decomposition

$$\frac{1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}.$$

Exercise 3: Solve for A, B, C in the "modified" partial fraction decomposition

$$\frac{1}{(s-1)(s^2-2s+10)} = \frac{A}{s-1} + \frac{B(s-1)+C}{(s-1)^2+9}.$$