Laplace transforms, transfer functions, and the impulse response formula

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Here, we shall focus on two aspects of the Laplace transform (LT):

- solving differential equations involving unit step (Heaviside) functions,
- convolutions and applications.

It follows from the definition of the LT that if

$$f(t) \stackrel{\mathcal{L}}{\longmapsto} F(s) = \mathcal{L}[f(t)](s),$$

then

$$f(t)u(t-c) \stackrel{\mathcal{L}}{\longmapsto} e^{-cs}\mathcal{L}[f(t+c)](s), \tag{1}$$

and

$$f(t-c)u(t-c) \stackrel{\mathcal{L}}{\longmapsto} e^{-cs}F(s). \tag{2}$$

These two properties are called *translation theorems.*

Example 1 *First, consider the Laplace transform of the piecewise-defined function $f(t) = (t-1)^2 u(t-1)$. Using (2), this is*

$$\mathcal{L}[f(t)] = e^{-s}\mathcal{L}[t^2](s) = 2\frac{1}{s^3}e^{-s}.$$

Second, consider the Laplace transform of the piecewise-constant function

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ -1 & \text{for } 0 \le t \le 2, \\ 1 & \text{for } t > 2. \end{cases}$$

This can be expressed as $f(t) = -u(t) + 2u(t-2)$, so

$$\begin{aligned} \mathcal{L}[f(t)] &= -\mathcal{L}[u(t)] + 2\mathcal{L}[u(t-2)] \\ &= -\frac{1}{s} + 2\frac{1}{s}e^{-2s}. \end{aligned}$$

Finally, consider the Laplace transform of $f(t) = \sin(t)u(t-\pi)$. Using (1) with $c = \pi$, this is

[1] The graphs were created using SAGE and and GIMP http://www.gimp.org/. Some of the latex code is taken from the excellent (public domain!) text by Sean Mauch [M]. Thanks to Darren Cruetz for comments and finding a serious typo. Last updated 2017-01-07.

$$\mathcal{L}[\sin(t)u(t-\pi)] = e^{-\pi s}\mathcal{L}[\sin(t+\pi)](s) = e^{-\pi s}\mathcal{L}[-\sin(t)](s) = -e^{-\pi s}\frac{1}{s^2+1},$$

thanks to the trig identity $\sin(t+\pi) = -\sin(t)$*. The plot of this function* $f(t) = \sin(t)u(t-\pi)$ *is displayed below:*

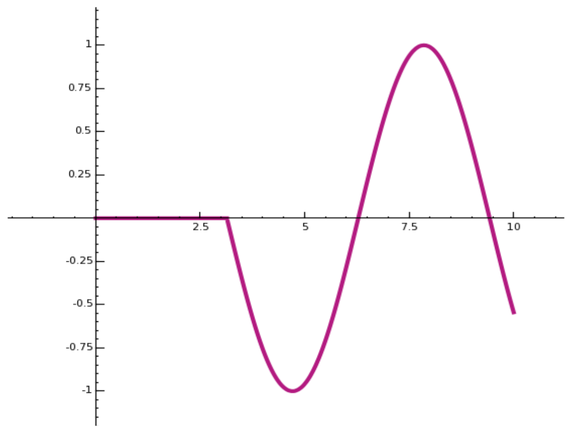

Figure 1: The piecewise continuous function $u(t-\pi)\sin(t)$.

We show how SAGE *can be used to compute these LTs.*

SAGE

```
sage: t = var('t')
sage: s = var('s')
sage: assume(s>0)
sage: f = Piecewise([[(0,1),0],[(1,infinity),(t-1)^2]])
sage: f.laplace(t, s)
2*e^(-s)/s^3
sage: f = Piecewise([[(0,2),-1],[(2,infinity),2]])
sage: f.laplace(t, s)
3*e^(-(2*s))/s - 1/s
sage: f = Piecewise([[(0,pi),0],[(pi,infinity),sin(t)]])
sage: f.laplace(t, s)
-e^(-(pi*s))/(s^2 + 1)
sage: f1 = lambda t: 0
sage: f2 = lambda t: sin(t)
sage: f = Piecewise([[(0,pi),f1],[(pi,10),f2]])
sage: P = f.plot(rgbcolor=(0.7,0.1,0.5),thickness=3)
sage: show(P)
```

The plot given by these last few commands is displayed above.

Before turning to differential equations, let us introduce convolutions.

Let $f(t)$ and $g(t)$ be continuous (for $t \geq 0$ - for $t < 0$, we assume $f(t) = g(t) = 0$). The *convolution* of $f(t)$ and $g(t)$ is defined by

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du = \int_0^t f(t-u)g(u)\,du = (g * f)(t).$$

(The equality between the integrals in the above equation is a result of a simple substitution, $u \to t - u$, which we leave to the reader.) The **convolution theorem** states

$$\mathcal{L}[(f * g)(t)](s) = F(s)G(s) = \mathcal{L}[f](s)\mathcal{L}[g](s). \tag{3}$$

In words: the LT of the convolution is the product of the LTs. (Or, equivalently, the inverse LT of the product is the convolution of the inverse LTs.)

To show this, do a change-of-variables in the following double integral:

$$\begin{aligned}
\mathcal{L}[f * g(t)](s) &= \int_0^\infty e^{-st} \int_0^t f(u)g(t-u)\,du\,dt \\
&= \int_0^\infty \int_u^\infty e^{-st} f(u)g(t-u)\,dt\,du \\
&= \int_0^\infty e^{-su} f(u) \int_u^\infty e^{-s(t-u)} g(t-u)\,dt\,du \\
&= \int_0^\infty e^{-su} f(u)\,du \int_0^\infty e^{-sv} g(v)\,dv \\
&= \mathcal{L}[f](s)\mathcal{L}[g](s).
\end{aligned}$$

Example 2 *Consider the inverse Laplace transform of* $\frac{1}{s^3 - s^2}$*. This can be computed using partial fractions and LT tables. However, it can also be computed using convolutions.*

First we factor the denominator, as follows

$$\frac{1}{s^3 - s^2} = \frac{1}{s^2}\frac{1}{s-1}.$$

We know the inverse Laplace transforms of each term:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t, \qquad \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = e^t$$

We apply the convolution theorem:

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-1}\right] &= \int_0^t u e^{t-u}\,du \\
&= e^t\left[-ue^{-u}\right]_0^t - e^t \int_0^t -e^{-u}\,du \\
&= -t - 1 + e^t
\end{aligned}$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-1}\right](t) = e^t - t - 1.$$

Example 3 *Here is a cool application of the convolution theorem. Consider the convolution*

$$f(t) = 1 * 1 * 1 * 1 * 1.$$

What is it? No one wants to compute a 5-tuple convolution directly from the integral definition. Here's an easier way. First, take the LT. Since the LT of the convolution is the product of the LTs:

$$\mathcal{L}[1 * 1 * 1 * 1 * 1](s) = (1/s)^5 = \frac{1}{s^5} = F(s).$$

Next, that the inverse LT. We know from LT tables that $\mathcal{L}^{-1}\left[\frac{4!}{s^5}\right](t) = t^4$*, so*

$$f(t) = \mathcal{L}^{-1}[F(s)](t) = \frac{1}{4!}\mathcal{L}^{-1}\left[\frac{4!}{s^5}\right](t) = \frac{1}{4!}t^4.$$

Now let us turn to solving a DE of the form

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_1. \tag{4}$$

First, take LTs of both sides:

$$as^2Y(s) - asy_0 - ay_1 + bsY(s) - by_0 + cY(s) = F(s),$$

so

$$Y(s) = \frac{1}{as^2 + bs + c}F(s) + \frac{asy_0 + ay_1 + by_0}{as^2 + bs + c}. \tag{5}$$

The function $\frac{1}{as^2+bs+c}$ is called the *transfer function* (this is an engineering term). Its inverse LT,

$$w(t) = \mathcal{L}^{-1}\left[\frac{1}{as^2 + bs + c}\right](t),$$

is sometimes called the *weight function* for the DE. (It's related to the Green's function discussed below.)

Lemma 4 *If* $a \neq 0$ *then* $w(0) = 0$.

(The only proof I have of this is a case-by-case proof using LT tables. Case 1 is when the roots of $as^2 + bs + c = 0$ are real and distinct, case 2 is when the roots are real and repeated, and case 3 is when the roots are complex. In each case, $w(0) = 0$. The verification of this is left to the reader, if he or she is interested.)

By the above lemma and the first derivative theorem,

$$w'(t) = \mathcal{L}^{-1}\left[\frac{s}{as^2+bs+c}\right](t).$$

Using this and the convolution theorem, the inverse LT of (5) gives the *impulse-response fomula*:

$$y(t) = (w * f)(t) + ay_0 \cdot w'(t) + (ay_1 + by_0) \cdot w(t). \tag{6}$$

This proves the following fact.

Theorem 5 *The unique solution to the DE (4) is (6).*

Example 6 *Consider the DE $y'' + y = 1$, $y(0) = y'(0) = 1$.*

The weight function is the inverse Laplace transform of $\frac{1}{s^2+1}$, so $w(t) = \sin(t)$. By (6),

$$y(t) = 1 * \sin(t) = \int_0^t \sin(u)\,du = -\cos(u)|_0^t = 1 - \cos(t).$$

(Yes, it is just that easy!)

If the "impulse" $f(t)$ is piecewise-defined, sometimes the convolution term in the formula (6) is awkward to compute.

Example 7 *Consider the DE $y'' - y' = u(t-1)$, $y(0) = y'(0) = 0$.*

Taking Laplace transforms gives $s^2Y(s) - sY(s) = \frac{1}{s}e^{-s}$, so

$$Y(s) = \frac{1}{s^3 - s^2}e^{-s}.$$

We know from a previous example that

$$\mathcal{L}^{-1}\left[\frac{1}{s^3 - s^2}\right](t) = e^t - t - 1,$$

so by the translation theorem (2), we have

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^3 - s^2}e^{-s}\right](t) = (e^{t-1} - (t-1) - 1) \cdot u(t-1) = (e^{t-1} - t) \cdot u(t-1).$$

At this stage, SAGE *lacks the functionality to solve this DE.*

Application to circuits: Consider an LRC circuit with applied voltage $v(t)$: the charge q on the capacitor satisfies the ODE

$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = v(t). \tag{7}$$

Taking Laplace transforms in equation (1) assuming $q(0) = q'(0) = 0$, we get

$$Ls^2Q(s) + RsQ(s) + \frac{1}{C}Q(s) = V(s)$$

or

$$Q(s) = \frac{V(s)}{Ls^2 + Rs + \frac{1}{C}}, \tag{8}$$

where $Q(s) = \mathcal{L}\{q(t)\}(s)$ and $V(s) = \mathcal{L}\{v(t)\}(s)$ are the Laplace transforms of $q(t)$ and $v(t)$ respectively. The function

$$G(s) = \frac{1}{Ls^2 + Rs + \frac{1}{C}} = \frac{C}{LCs^2 + RCs + 1}$$

is the *transfer function* of the system. If we let $g(t) = \mathcal{L}^{-1}\{G(s)\}(t)$ be its inverse transform, then taking inverse transforms in equation (2) and applying the convolution theorem gives

$$q(t) = g(t) * v(t) = \int_0^t g(t-w)v(w)dw. \tag{9}$$

This equation says that the response of the system to any applied voltage $v(t)$ can be obtained by integrating $v(w)$ against $g(t-w)$ (The function $\mathcal{G}(t, w) = g(t-w)$ of two variables is called the *Green's function*[2] for this situation).

The function $g(t)$ is sometimes called the *impulse response* for the following reason. Suppose the input function $v(t)$ is the Dirac delta function. Then equation (3) gives

$$q(t) = \int_0^t g(t-w)\delta(w)dw = g(t).$$

That is, $g(t)$ is the response of the system when forced by the Dirac delta (impulse) function.

[2]The Green's function is named after George Green (1793-1841), a British mathematician, who also discovered the (unrelated) result called Green's Theorem from vector calculus.

Exercise 1

1. Suppose a mass m hangs from a spring with spring constant k, and is subject to a damping force equal to β times its velocity and an external force $f(t)$. Then the distance $x(t)$ of the mass below equilibrium satisfies the differential equation

$$mx''(t) + \beta x'(t) + kx(t) = f(t)$$

by Newton's second law.
(a) What is the transfer function in this case?
(b) Suppose $m = 1$ kg, $\beta = 2$ N-sec/m, and $k = 5$ N/m. Assuming $x(0) = x'(0) = 0$, write $x(t)$ as a convolution using the impulse-response fomula.

Exercise 2: (a) Use SAGE to take the LT of $u(t - \pi/4)\cos(t)$.
(b) Use SAGE to compute the convolution $\sin(t) * \cos(t)$.

References

[L] Wikipedia entry for Laplace: http://en.wikipedia.org/wiki/Pierre-Simon_Laplace

[LT] Wikipedia entry for Laplace transform: http://en.wikipedia.org/wiki/Laplace_transform

[M] Sean Mauch, *Introduction to methods of Applied Mathematics*, http://www.its.caltech.edu/~sean/book/unabridged.html