Laplace transforms, transfer functions, and the impulse response formula

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Here, we shall focus on two aspects of the Laplace transform (LT):

- solving differential equations involving unit step (Heaviside) functions,
- convolutions and applications.

It follows from the definition of the LT that if

\[ f(t) \xrightarrow{\mathcal{L}} F(s) = \mathcal{L}[f(t)](s), \]

then

\[ f(t)u(t-c) \xrightarrow{\mathcal{L}} e^{-cs}F(s), \] (1)

and

\[ f(t-c)u(t-c) \xrightarrow{\mathcal{L}} e^{-cs}L[f(t+c)](s). \] (2)

These two properties are called translation theorems.

Example 1 First, consider the Laplace transform of the piecewise-defined function \( f(t) = (t-1)^2u(t-1) \). Using (2), this is

\[ \mathcal{L}[f(t)] = e^{-s}s^3 - \frac{1}{s^3}e^{-s}. \]

Second, consider the Laplace transform of the piecewise-constant function

\[ f(t) = \begin{cases} 
0 & \text{for } t < 0, \\
-1 & \text{for } 0 \leq t \leq 2, \\
1 & \text{for } t > 2.
\end{cases} \]

This can be expressed as \( f(t) = -u(t) + 2u(t-2) \), so

\[ \mathcal{L}[f(t)] = -L[u(t)] + 2L[u(t-2)] 
= \frac{1}{s} + 2\frac{1}{s}e^{-2s}. \]

Finally, consider the Laplace transform of \( f(t) = \sin(t)u(t-\pi) \). Using (1) with \( c = \pi \), this is

\[ \mathcal{L}[f(t)] = -\mathcal{L}[u(t)] + 2\mathcal{L}[u(t-\pi)] 
= \frac{1}{s} + 2\frac{1}{s}e^{-2s}. \]

1 These notes licensed under Attribution-ShareAlike Creative Commons license, http://creativecommons.org/about/licenses/meet-the-licenses The graphs were created using SAGE and GIMP http://www.gimp.org/ Some of the latex code is taken from the excellent (public domain!) text by Sean Mauch [M]. Thanks to Darren Cruetz for comments and finding a serious typo. Last updated 2017-01-07.
\[\mathcal{L}[\sin(t)u(t-\pi)] = e^{-\pi s} \mathcal{L}[\sin(t+\pi)](s) = e^{-\pi s} \mathcal{L}[-\sin(t)](s) = \frac{1}{s^2 + 1},\]

thanks to the trig identity \(\sin(t+\pi) = -\sin(t)\). The plot of this function \(f(t) = \sin(t)u(t-\pi)\) is displayed below:

![Graph of \(\sin(t)u(t-\pi)\)](image)

Figure 1: The piecewise continuous function \(u(t-\pi)\sin(t)\).

We show how SAGE can be used to compute these LTs.

```sage
sage: t = var('t')
sage: s = var('s')
sage: assume(s>0)
sage: f = Piecewise([[0,1],0],[(1,infinity),(t-1)^2])
sage: f.laplace(t, s)
2*e^(-s)/s^3
sage: f = Piecewise([[0,2],[-1],[2,infinity],[2]])
sage: f.laplace(t, s)
3*e^(-2s)/s - 1/s
sage: f = Piecewise([[0,pi],0],[(pi,infinity),\sin(t)])
sage: f.laplace(t, s)
-e^(-pi s)/(s^2 + 1)
sage: f1 = lambda t: 0
sage: f2 = lambda t: \sin(t)
sage: f = Piecewise([[0,pi],[f1],[pi,10],[f2]])
sage: P = f.plot(rgbcolor=(0.7,0.1,0.5),thickness=3)
sage: show(P)
```

The plot given by these last few commands is displayed above.

Before turning to differential equations, let us introduce convolutions.
Let \( f(t) \) and \( g(t) \) be continuous (for \( t \geq 0 \) - for \( t < 0 \), we assume \( f(t) = g(t) = 0 \)). The convolution of \( f(t) \) and \( g(t) \) is defined by

\[
(f * g)(t) = \int_0^t f(u) g(t-u) \, du = \int_0^t f(t-u) g(u) \, du = (g * f)(t).
\]

(The equality between the integrals in the above equation is a result of a simple substitution, \( u \to t-u \), which we leave to the reader.) The convolution theorem states

\[
\mathcal{L}[(f * g)(t)](s) = F(s)G(s) = \mathcal{L}[f](s)\mathcal{L}[g](s). \tag{3}
\]

In words: the LT of the convolution is the product of the LTs. (Or, equivalently, the inverse LT of the product is the convolution of the inverse LTs.)

To show this, do a change-of-variables in the following double integral:

\[
\mathcal{L}[f * g(t)](s) = \int_0^\infty e^{-st} \int_0^t f(u) g(t-u) \, du \, dt
\]

\[
= \int_0^\infty \int_u^\infty e^{-st} f(u) g(t-u) \, dt \, du
\]

\[
= \int_0^\infty e^{-su} f(u) \int_u^\infty e^{-s(t-u)} g(t-u) \, dt \, du
\]

\[
= \int_0^\infty e^{-su} f(u) \int_0^\infty e^{-sv} g(v) \, dv
\]

\[
= \mathcal{L}[f](s)\mathcal{L}[g](s).
\]

**Example 2** Consider the inverse Laplace transform of \( \frac{1}{s^3 - s^2} \). This can be computed using partial fractions and LT tables. However, it can also be computed using convolutions.

First we factor the denominator, as follows

\[
\frac{1}{s^3 - s^2} = \frac{1}{s^2} \frac{1}{s - 1}.
\]

We know the inverse Laplace transforms of each term:

\[
\mathcal{L}^{-1} \left[ \frac{1}{s^2} \right] = t, \quad \mathcal{L}^{-1} \left[ \frac{1}{s - 1} \right] = e^t
\]

We apply the convolution theorem:

\[
\mathcal{L}^{-1} \left[ \frac{1}{s^2} \frac{1}{s - 1} \right] = \int_0^t u e^{t-u} \, du
\]

\[
= e^t \left[ -ue^{-u} \right]_0^t - e^t \int_0^t -e^{-u} \, du
\]

\[
= -t - 1 + e^t
\]
Therefore,
\[ \mathcal{L}^{-1} \left[ \frac{1}{s^2} - \frac{1}{s-1} \right] (t) = e^t - t - 1. \]

**Example 3** Here is a cool application of the convolution theorem. Consider the convolution
\[ f(t) = 1 * 1 * 1 * 1 * 1. \]
What is it? No one wants to compute a 5-tuple convolution directly from the integral definition. Here’s an easier way. First, take the LT. Since the LT of the convolution is the product of the LTs:
\[ \mathcal{L}[1 * 1 * 1 * 1 * 1](s) = \frac{1}{s^5} = F(s). \]
Next, that the inverse LT. We know from LT tables that \( \mathcal{L}^{-1} \left[ \frac{4!}{s^5} \right] (t) = t^4, \) so
\[ f(t) = \mathcal{L}^{-1} [F(s)] (t) = \frac{1}{4!} \mathcal{L}^{-1} \left[ \frac{4!}{s^5} \right] (t) = \frac{1}{4!} t^4. \]

Now let us turn to solving a DE of the form
\[ ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_1. \] (4)
First, take LTs of both sides:
\[ as^2Y(s) - asy_0 - ay_1 + bsY(s) - by_0 + cY(s) = F(s), \]
so
\[ Y(s) = \frac{1}{as^2 + bs + c} F(s) + \frac{asy_0 + ay_1 + by_0}{as^2 + bs + c}. \] (5)
The function \( \frac{1}{as^2 + bs + c} \) is called the **transfer function** (this is an engineering term). Its inverse LT,
\[ w(t) = \mathcal{L}^{-1} \left[ \frac{1}{as^2 + bs + c} \right] (t), \]
is sometimes called the **weight function** for the DE. (It’s related to the Green’s function discussed below.)

**Lemma 4** If \( a \neq 0 \) then \( w(0) = 0. \)

(The only proof I have of this is a case-by-case proof using LT tables. Case 1 is when the roots of \( as^2 + bs + c = 0 \) are real and distinct, case 2 is when the roots are real and repeated, and case 3 is when the roots are complex. In each case, \( w(0) = 0. \) The verification of this is left to the reader, if he or she is interested.)
By the above lemma and the first derivative theorem,

\[ w'(t) = \mathcal{L}^{-1}\left[ \frac{s}{as^2 + bs + c} \right](t). \]

Using this and the convolution theorem, the inverse LT of \(5\) gives the impulse-response formula:

\[ y(t) = (w * f)(t) + ay_0 \cdot w'(t) + (ay_1 + by_0) \cdot w(t). \]  

(6)

This proves the following fact.

**Theorem 5** The unique solution to the DE \((4)\) is \((6)\).

**Example 6** Consider the DE \(y'' + y = 1, \quad y(0) = y'(0) = 1\).

The weight function is the inverse Laplace transform of \(\frac{1}{s^2+1}\), so \(w(t) = \sin(t)\). By \(6\),

\[ y(t) = 1 * \sin(t) = \int_0^t \sin(u) du = -\cos(u)|^t_0 = 1 - \cos(t). \]

(Yes, it is just that easy!)

If the “impulse” \(f(t)\) is piecewise-defined, sometimes the convolution term
in the formula \(6\) is awkward to compute.

**Example 7** Consider the DE \(y'' - y' = u(t - 1), \quad y(0) = y'(0) = 0\).

Taking Laplace transforms gives \(s^2Y(s) - sY(s) = \frac{1}{s}e^{-s}\), so

\[ Y(s) = \frac{1}{s^3 - s^2}e^{-s}. \]

We know from a previous example that

\[ \mathcal{L}^{-1}\left[ \frac{1}{s^3 - s^2} \right](t) = e^t - t - 1, \]

so by the translation theorem \(3\), we have

\[ y(t) = \mathcal{L}^{-1}\left[ \frac{1}{s^3 - s^2}e^{-s} \right](t) = (e^{t-1} - (t-1) - 1) \cdot u(t-1) = (e^{t-1} - t) \cdot u(t-1). \]

At this stage, SAGE lacks the functionality to solve this DE.

**Application to circuits**: Consider an LRC circuit with applied voltage \(v(t)\): the charge \(q\) on the capacitor satisfies the ODE

\[ Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = v(t). \]  

(7)
Taking Laplace transforms in equation (1) assuming \( q(0) = q'(0) = 0 \), we get

\[ Ls^2 Q(s) + RsQ(s) + \frac{1}{C} Q(s) = V(s) \]

or

\[ Q(s) = \frac{V(s)}{Ls^2 + Rs + \frac{1}{C}}, \tag{8} \]

where \( Q(s) = \mathcal{L}\{q(t)\}(s) \) and \( V(s) = \mathcal{L}\{v(t)\}(s) \) are the Laplace transforms of \( q(t) \) and \( v(t) \) respectively. The function

\[ G(s) = \frac{C}{LCs^2 + RCs + 1} \]

is the transfer function of the system. If we let \( g(t) = \mathcal{L}^{-1}\{G(s)\}(t) \) be its inverse transform, then taking inverse transforms in equation (2) and applying the convolution theorem gives

\[ q(t) = g(t) \ast v(t) = \int_0^t g(t - w)v(w)dw. \tag{9} \]

This equation says that the response of the system to any applied voltage \( v(t) \) can be obtained by integrating \( v(w) \) against \( g(t - w) \) (The function \( G(t, w) = g(t - w) \) of two variables is called the Green’s function for this situation).

The function \( g(t) \) is sometimes called the impulse response for the following reason. Suppose the input function \( v(t) \) is the Dirac delta function. Then equation (3) gives

\[ q(t) = \int_0^t g(t - w)\delta(w)dw = g(t). \]

That is, \( g(t) \) is the response of the system when forced by the Dirac delta (impulse) function.

\[ ^2 \text{The Green’s function is named after George Green (1793-1841), a British mathematician, who also discovered the (unrelated) result called Green’s Theorem from vector calculus.} \]
Exercise 1

1. Suppose a mass $m$ hangs from a spring with spring constant $k$, and is subject to a damping force equal to $\beta$ times its velocity and an external force $f(t)$. Then the distance $x(t)$ of the mass below equilibrium satisfies the differential equation

$$mx''(t) + \beta x'(t) + kx(t) = f(t)$$

by Newton’s second law.

(a) What is the transfer function in this case?
(b) Suppose $m = 1$ kg, $\beta = 2$ N-sec/m, and $k = 5$ N/m. Assuming $x(0) = x'(0) = 0$, write $x(t)$ as a convolution using the impulse-response formula.

Exercise 2: (a) Use SAGE to take the LT of $u(t - \pi/4)\cos(t)$.
(b) Use SAGE to compute the convolution $\sin(t) \ast \cos(t)$.

References

[L] Wikipedia entry for Laplace:  
\url{http://en.wikipedia.org/wiki/Pierre-Simon_Laplace}

[LT] Wikipedia entry for Laplace transform:  
\url{http://en.wikipedia.org/wiki/Laplace_transform}

\url{http://www.its.caltech.edu/~sean/book/unabridged.html}