Introduction to variation of parameters for systems

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The method called variation of parameters for *systems* of ODEs has no relation to the method variation of parameters for 2nd order ODEs discussed in an earlier lecture except for their name.

Early background:

Recall that when we solved the 1st order ODE

$$y' = ay, \qquad y(0) = y_0, \tag{1}$$

for $y = y(t)$ using the method of separation of variables, we got the formula

$$y = ce^{at} = e^{at}c, \tag{2}$$

where c is a constant depending on the initial condition (in fact, $c = y(0)$).

Consider a 2×2 system of linear 1st order ODEs in the form

$$\begin{cases} x' = ax + by), & x(0) = x_0, \\ y' = cx + dy, & y(0) = y_0. \end{cases}$$

This can be rewritten in the form

$$\vec{X}' = A\vec{X}, \tag{3}$$

where $\vec{X} = \vec{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, and A is the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We can solve (3) analogously to (1), to obtain

$$\vec{X} = e^{tA}\vec{c}, \tag{4}$$

$\vec{c} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$ is a constant depending on the initial conditions and e^{tA} is a "matrix exponential" defined by

$$e^B = \sum_{n=0}^{\infty} \frac{1}{n!} B^n,$$

for any square matrix B.

You might be thinking to yourself: I can't compute the matrix exponential so what good is this formula (4)? Good question! The answer is that the eigenvalues and eigenvectors of the matrix A enable you to compute e^{tA}. This is the basis for the formulas for the solution of a system of ODEs using the eigenvalue method which you encountered in an earlier lecture.

The eigenvalue method showed that every solution to (3) could be written in the form

$$\vec{X} = c_1\vec{X}_1(t) + c_2\vec{X}_2(t),$$

for some vector-valued solutions $\vec{X}_1(t)$ and $\vec{X}_2(t)$ called **fundamental solutions**. Frequently, we call the matrix of fundamental solutions,

$$\Phi = \left(\vec{X}_1(t), \vec{X}_2(t)\right),$$

the **fundamental matrix**. The fundamental matrix is, roughly speaking, e^{tA}.

See examples below for more details.

Basic idea:

Recall that we we solved the 1st order ODE

$$y' + p(t)y = q(t) \tag{5}$$

for $y = y(t)$ using the method of integrating factors, we got the formula

$$y = (e^{\int p(t)\,dt})^{-1}(\int e^{\int p(t)\,dt} q(t)\,dt \ + \ c). \tag{6}$$

Consider a 2×2 system of linear 1st order ODEs in the form

$$\begin{cases} x' = ax + by + f(t), & x(0) = x_0, \\ y' = cx + dy + g(t), & y(0) = y_0. \end{cases}$$

This can be rewritten in the form

$$\vec{X}' = A\vec{X} + \vec{F}, \tag{7}$$

where $\vec{F} = \vec{F}(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$. Equation (7) can be seen to be in a form analogous to (5) by replacing $\vec{X}$ by y, A by $-p$ and $\vec{F}$ by q. It turns out that (7) can be solved in a way analogous to (5) as well. Here is the formula:

$$\vec{X} = \Phi(\int \Phi^{-1}\vec{F}(t)\,dt \; + \; \vec{c}), \tag{8}$$

where $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is a constant vector determined by the initial conditions and Φ is the fundamental matrix.

Example 1 A battle between X-men and Y-men is modeled by

$$\begin{cases} x' = -y + 1, & x(0) = 100, \\ y' = -4x + e^t, & y(0) = 50. \end{cases}$$

The non-homogeneous terms 1 and e^t represent reinforcements. Find out who wins, when, and the number of survivers.

Here A is the matrix

$$A = \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix}$$

and $\vec{F} = \vec{F}(t) = \begin{pmatrix} 1 \\ e^t \end{pmatrix}$.

In the method of variation of parameters, you must solve the homogeneous system first.

The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = -2$, with associated eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, resp.. The general solution to the homogeneous system

$$\begin{cases} x' = -y, \\ y' = -4x, \end{cases}$$

is

$$\vec{X} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} = c_1\vec{X}_1(t) + c_2\vec{X}_2(t),$$

where

$$\vec{X}_1(t) = \begin{pmatrix} e^{2t} \\ -2e^{2t} \end{pmatrix}, \qquad \vec{X}_2(t) = \begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}.$$

For the solution of the non-homogeneous equation, we must compute the fundamental matrix:

$$\Phi = \begin{pmatrix} e^{2t} & e^{-2t} \\ -2e^{2t} & 2e^{-2t} \end{pmatrix}, \quad \text{so} \quad \Phi^{-1} = \frac{1}{4}\begin{pmatrix} 2e^{-2t} & -e^{-2t} \\ 2e^{2t} & e^{2t} \end{pmatrix}.$$

Next, we compute the product,

$$\Phi^{-1}\vec{F} = \frac{1}{4}\begin{pmatrix} 2e^{-2t} & -e^{-2t} \\ 2e^{2t} & e^{2t} \end{pmatrix}\begin{pmatrix} 1 \\ e^t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-2t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{2t} + \frac{1}{4}e^{3t} \end{pmatrix}$$

and its integral,

$$\int \Phi^{-1}\vec{F}\,dt = \begin{pmatrix} -\frac{1}{4}e^{-2t} + \frac{1}{4}e^{-t} \\ \frac{1}{4}e^{2t} + \frac{1}{12}e^{3t} \end{pmatrix}.$$

Finally, to finish (8), we compute

$$\begin{aligned} \Phi(\textstyle\int \Phi^{-1}\vec{F}(t)\,dt \ + \ \vec{c}) &= \begin{pmatrix} e^{-2t} & e^{-2t} \\ -2e^{2t} & 2e^{2t} \end{pmatrix}\begin{pmatrix} -\frac{1}{4}e^{-2t} + \frac{1}{4}e^{-t} + c1 \\ \frac{1}{4}e^{2t} + \frac{1}{12}e^{3t} + c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_1e^{2t} + \frac{1}{3}e^t + c_2e^{-2t} \\ 1 - \frac{1}{3}e^t - 2c_1e^{2t} + 2c_2e^{-2t} \end{pmatrix}. \end{aligned}$$

This gives the general solution to the original system

$$x(t) = c_1e^{2t} + \frac{1}{3}e^t + c_2e^{-2t},$$

and

$$y(t) = 1 - \frac{1}{3}e^t - 2c_1e^{2t} + 2c_2e^{-2t}.$$

We aren't done! It remains to compute c_1, c_2 using the ICs. For this, solve

$$\frac{1}{3} + c_1 + c_2 = 100, \qquad \frac{2}{3} - 2c_1 + 2c_2 = 50.$$

We get

$$c_1 = 75/2, \qquad c_2 = 373/6,$$

so

$$x(t) = \frac{75}{2}e^{2t} + \frac{1}{3}e^{t} + \frac{373}{6}e^{-2t},$$

and

$$y(t) = 1 - \frac{1}{3}e^{t} - 75e^{2t} + \frac{373}{3}e^{-2t}.$$

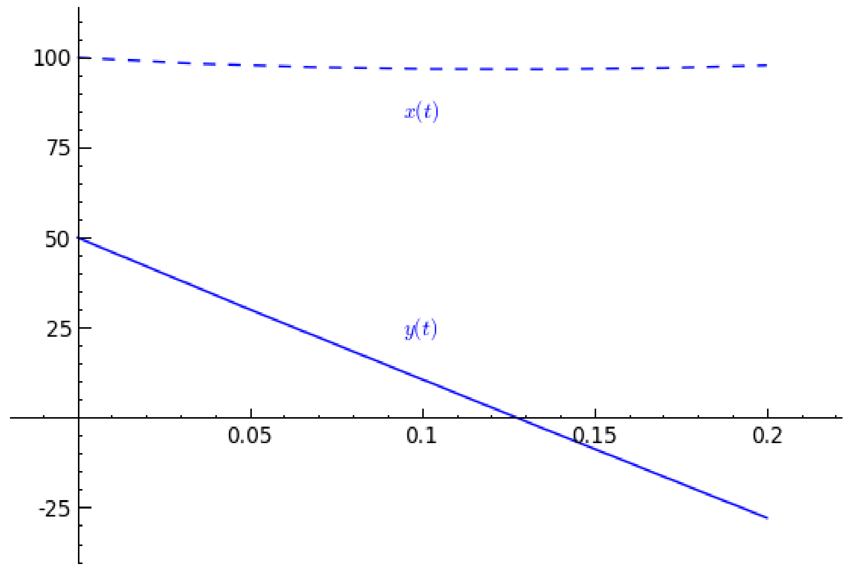

Figure 1: Solution to system $x' = -y + 1$, $x(0) = 100$, $y' = -4x + e^t$, $y(0) = 50$.

As you can see from Figure 1, the X-men win. The solution to $y(t) = 0$ is about $t_0 = 0.1279774...$ and $x(t_0) = 96.9458...$ "survive".