Undetermined coefficients in constant coefficient ODEs

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The method of undetermined coefficients [U] can be used to solve the following type of problem.

PROBLEM: Solve

$$ay'' + by' + cy = f(x), \tag{1}$$

where $a \neq 0$, b and c are constants. (Even the case $a = 0$ can be handled similarly, though some of the discussion below might need to be slightly modified.) For this method to work, we must assume that $f(x)$ is of a special form (described below). The version of this method described below can be found in Spiegel's (long out-of-print) textbook [S] and requires no memorization.

More-or-less equivalent is the method of annihilating operators [A] (they solve the same class of DEs), but that method will be discussed separately.

For the moment, let us assume $f(x)$ has the form $a_1 \cdot p(x) \cdot e^{a_2 x} \cdot \cos(a_3 x)$, or $a_1 \cdot p(x) \cdot e^{a_2 x} \cdot \sin(a_3 x)$, where a_1, a_2, a_3 are constants and $p(x)$ is a polynomial. (If $f(x)$ is a sum of such functions, then fisrt solve the DEs for each of the "parts" of $f(x)$ then add them up[2].)

Solution:

- Find the "homogeneous solution" y_h to $ay'' + by' + cy = 0$, $y_h = c_1 y_1 + c_2 y_2$. Here y_1 and y_2 are determined as follows: let r_1 and r_2 denote the roots of the characteristic polynomial $aD^2 + bD + c = 0$.
  - $r_1 \neq r_2$ real: set $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$.
  - $r_1 = r_2$ real: if $r = r_1 = r_2$ then set $y_1 = e^{rx}$, $y_2 = xe^{rx}$.

[2]By the superposition principle (or "linearity"), if y_1 is a solution to $ay'' + by' + cy = f_1(x)$ and y_2 is a solution to $ay'' + by' + cy = f_2(x)$, then $y_1 + y_2$ is a solution to $ay'' + by' + cy = f_1(x) + f_2(x)$.

- r_1, r_2 complex: if $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, where α and β are real, then set $y_1 = e^{\alpha x}\cos(\beta x)$, $y_2 = e^{\alpha x}\sin(\beta x)$.

- Compute $f(x)$, $f'(x)$, $f''(x)$, ... Write down the list of all the different terms which arise (via the product rule), ignoring constant factors, plus signs, and minus signs:

$$f_1(x),\ f_2(x),\ \ldots,\ \ f_k(x).$$

If any one of these agrees with y_1 or y_2 then multiply them all by x. (If, after this, any of them *still* agrees with y_1 or y_2 then multiply them all again by x.)

- Let y_p be a linear combination of these functions (your "guess"):

$$y_p = A_1 f_1(x) + \ldots + A_k f_k(x).$$

This is called the **general form of the particular solution**. The A_i's are called **undetermined coefficients**.

- Plug y_p into (1) and solve for A_1, ..., A_k.

- Let $y = y_h + y_p = y_p + c_1 y_1 + c_2 y_2$. This is the **general solution** to (1). If there are any initial conditions for (1), solve for then c_1, c_2 now.

Diagramatically:

Factor characteristic polynomial

↓

Compute y_h

↓

Compute the general form of the particular, y_p

↓

Compute the undetermined coefficients

$$\downarrow$$

$$\boxed{\text{Answer: } y = y_h + y_p.}$$

Examples

Example 1 *Solve*

$$y'' - y = \cos(2x).$$

- *The characteristic polynomial is* $r^2 - 1 = 0$*, which has* ± 1 *for roots. The "homogeneous solution" is therefore* $y_h = c_1 e^x + c_2 e^{-x}$.
- *We compute* $f(x) = \cos(2x)$, $f'(x) = -2\sin(2x)$, $f''(x) = -4\cos(2x)$, ... *They are all linear combinations of*

  $$f_1(x) = \cos(2x),\ f_2(x) = \sin(2x).$$

  None of these agrees with $y_1 = e^x$ *or* $y_2 = e^{-x}$*, so we do not multiply by* x.
- *Let* y_p *be a linear combination of these functions:*

  $$y_p = A_1 \cos(2x) + A_2 \sin(2x).$$

- *You can compute both sides of* $y_p'' - y_p = \cos(2x)$:

  $$(-4A_1 \cos(2x) - 4A_2 \sin(2x)) - (A_1 \cos(2x) + A_2 \sin(2x)) = \cos(2x).$$

  Equating the coefficients of $\cos(2x)$, $\sin(2x)$ *on both sides gives 2 equations in 2 unknowns:* $-5A_1 = 1$ *and* $-5A_2 = 0$. *Solving, we get* $A_1 = -1/5$ *and* $A_2 = 0$.
- *The general solution:* $y = y_h + y_p = c_1 e^x + c_2 e^{-x} - \frac{1}{5}\cos(2x)$.

Example 2 *Solve*

$$y'' - y = x\cos(2x).$$

- *The characteristic polynomial is* $r^2 - 1 = 0$*, which has* ± 1 *for roots. The "homogeneous solution" is therefore* $y_h = c_1 e^x + c_2 e^{-x}$.
- *We compute* $f(x) = x\cos(2x)$, $f'(x) = \cos(2x) - 2x\sin(2x)$, $f''(x) = -2\sin(2x) - 2\sin(2x) - 2x\cos(2x)$, ... *They are all linear combinations of*

  $$f_1(x) = \cos(2x),\ f_2(x) = \sin(2x),\ f_3(x) = x\cos(2x),\ .f_4(x) = x\sin(2x).$$

  None of these agrees with $y_1 = e^x$ *or* $y_2 = e^{-x}$*, so we do not multiply by* x.

- *Let y_p be a linear combination of these functions:*

$$y_p = A_1 \cos(2x) + A_2 \sin(2x) + A_3 x \cos(2x) + A_4 x \sin(2x).$$

- *In principle, you can compute both sides of $y_p'' - y_p = x\cos(2x)$ and solve for the A_i's. (Equate coefficients of $x\cos(2x)$ on both sides, equate coefficients of $\cos(2x)$ on both sides, equate coefficients of $x\sin(2x)$ on both sides, and equate coefficients of $\sin(2x)$ on both sides. This gives 4 equations in 4 unknowns, which can be solved.) You will not be asked to solve for the A_i's for a problem this hard.*

Example 3 *Solve*

$$y'' + 4y = x\cos(2x).$$

- *The characteristic polynomial is $r^2 + 4 = 0$, which has $\pm 2i$ for roots. The "homogeneous solution" is therefore $y_h = c_1 \cos(2x) + c_2 \sin(2x)$.*

- *We compute $f(x) = x\cos(2x)$, $f'(x) = \cos(2x) - 2x\sin(2x)$, $f''(x) = -2\sin(2x) - 2\sin(2x) - 2x\cos(2x)$, ... They are all linear combinations of*

$$f_1(x) = \cos(2x),\ f_2(x) = \sin(2x),\ f_3(x) = x\cos(2x),\ .f_4(x) = x\sin(2x).$$

  Two of these agree with $y_1 = \cos(2x)$ or $y_2 = \sin(2x)$, so we <u>do</u> multiply by x:

$$f_1(x) = x\cos(2x),\ f_2(x) = x\sin(2x),\ f_3(x) = x^2\cos(2x),\ .f_4(x) = x^2\sin(2x).$$

- *Let y_p be a linear combination of these functions:*

$$y_p = A_1 x\cos(2x) + A_2 x\sin(2x) + A_3 x^2 \cos(2x) + A_4 x^2 \sin(2x).$$

- *In principle, you can compute both sides of $y_p'' + 4y_p = x\cos(2x)$ and solve for the A_i's. You will not be asked to solve for the A_i's for a problem this hard.*

More generally, suppose that you want to solve $ay'' + by' + cy = f(x)$, where $f(x)$ is a *sum* of functions of the above form. In other words, $f(x) = f_1(x) + f_2(x) + ... + f_k(x)$, where each $f_j(x)$ is of the form $c \cdot p(x) \cdot e^{ax} \cdot \cos(bx)$, or $c \cdot p(x) \cdot e^{ax} \cdot \sin(bx)$, where a, b, c are constants and $p(x)$ is a polynomial. You can proceed in either one of the following ways.

1. Split up the problem by solving each of the k problems $ay'' + by' + cy = f_j(x)$, $1 \le j \le k$, obtaining the solution $y = y_j(x)$, say. The solution to $ay'' + by' + cy = f(x)$ is then $y = y_1 + y_2 + .. + y_k$ (the superposition principle).

2. Proceed as in the examples above but with the following slight revision:

   - Find the "homogeneous solution" y_h to $ay'' + by' = cy = 0$, $y_h = c_1 y_1 + c_2 y_2$.
   - Compute $f(x)$, $f'(x)$, $f''(x)$, ... Write down the list of all the different terms which arise, ignoring constant factors, plus signs, and minus signs:

   $$t_1(x),\ t_2(x),\ ...,\ \ t_k(x).$$

   - Group these terms into their *families*. Each family is determined from its parent(s) - which are the terms in $f(x) = f_1(x) + f_2(x) + ... + f_k(x)$ which they arose form by differentiation. For example, if $f(x) = x\cos(2x) + e^{-x}\sin(x) + \sin(2x)$ then the terms you get from differentiating and ignoring constants, plus signs and minus signs, are

   $$x\cos(2x), x\sin(2x), \cos(2x), \sin(2x), \qquad (\text{from } x\cos(2x)),$$

   $$e^{-x}\sin(x), e^{-x}\cos(x), \qquad (\text{from } e^{-x}\sin(x)),$$

   and

   $$\sin(2x), \cos(2x), \qquad (\text{from } \sin(2x)).$$

   The first group absorbes the last group, since you can only count the *different* terms. Therefore, there are only two families in this example: $\{x\cos(2x), x\sin(2x), \cos(2x), \sin(2x)\}$ is a "family" (with "parent" $x\cos(2x)$ and the other terms as its "children") and $\{e^{-x}\sin(x), e^{-x}\cos(x)\}$ is a "family" (with "parent" $e^{-x}\sin(x)$ and the other term as its "child").

   If any one of these terms agrees with y_1 or y_2 then multiply the *entire family* by x. In other words, if any child or parent is "bad" then the entire family is "bad". (If, after this, any of them *still* agrees with y_1 or y_2 then multiply them all again by x.)

- Let y_p be a linear combination of these functions (your "guess"):

$$y_p = A_1 t_1(x) + ... + A_k t_k(x).$$

  This is called the **general form of the particular solution**. The A_i's are called **undetermined coefficients**.

- Plug y_p into (1) and solve for A_1, ..., A_k.
- Let $y = y_h + y_p = y_p + c_1 y_1 + c_2 y_2$. This is the **general solution** to (1). If there are any initial conditions for (1), solve for then c_1, c_2 *last - after the undetermined coefficients.*

Example 4 *Solve*

$$y''' - y'' - y' + y = 12xe^x.$$

We use SAGE *for this.*

SAGE

```
sage: x = var("x")
sage: y = function("y",x)
sage: R.<D> = PolynomialRing(QQ, "D")
sage: f = D^3 - D^2 - D + 1
sage: f.factor()
 (D + 1) * (D - 1)^2
sage: f.roots()
 [(-1, 1), (1, 2)]
```

So the roots of the characteristic polynomial are $1, 1, -1$*, which means that the homogeneous part of the solution is*

$$y_h = c_1 e^x + c_2 e^{-x} + c_3 x e^{-x}.$$

SAGE

```
sage: de = lambda y: diff(y,x,3) - diff(y,x,2) - diff(y,x,1) + y
sage: c1 = var("c1"); c2 = var("c2"); c3 = var("c3")
sage: yh = c1*e^x + c2*x*e^x + c3*e^(-x)
```

```
sage: de(yh)
 0
sage: de(x^3*e^x-(3/2)*x^2*e^x)
 12*x*e^x
```

This just confirmed that y_h solves $y''' - y'' - y' + y = 0$. Using the derivatives of $F(x) = 12xe^x$, we generate the general form of the particular:

SAGE

```
sage: F = 12*x*e^x
sage: diff(F,x,1); diff(F,x,2); diff(F,x,3)
 12*x*e^x + 12*e^x
 12*x*e^x + 24*e^x
 12*x*e^x + 36*e^x
sage: A1 = var("A1"); A2 = var("A2")
sage: yp = A1*x^2*e^x + A2*x^3*e^x
```

Now plug this into the DE and compare coefficients of like terms to solve for the undertermined coefficients:

SAGE

```
sage: de(yp)
 12*x*e^x*A2 + 6*e^x*A2 + 4*e^x*A1
sage: solve([12*A2 == 12, 6*A2+4*A1 == 0],A1,A2)
 [[A1 == -3/2, A2 == 1]]
```

Finally, lets check if this is correct:

SAGE

```
sage: y = yh + (-3/2)*x^2*e^x + (1)*x^3*e^x
sage: de(y)
 12*x*e^x
```

Exercise: Using SAGE, solve

$$y''' - y'' + y' - y = 12xe^x.$$

References

[A] Wikipedia entry for the annihilator method: http://en.wikipedia.org/wiki/Annihilator_method

[S] Murray R. Spiegel, *Applied differential equations*, Prentice Hall. (out of print textbook)

[U] General wikipedia introduction to undetermined coefficients: http://en.wikipedia.org/wiki/Method_of_undetermined_coefficients