Solving ODEs:
using the power series method, II

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In this part, we solve some DEs using power series.

We want to solve a problem of the form

$$y''(x) + p(x)y'(x) + y(x) = f(x), \tag{1}$$

in the case where $p(x)$, $q(x)$ and $f(x)$ have a power series expansion. We will call a **power series solution** a series expansion for $y(x)$ where we have produced some algorithm or rule which enables us to compute as many of its coefficients as we like.

Solution strategy: Write $y(x) = a_0 + a_1x + a_2x^2 + ... = \sum_{k=0}^{\infty} a_k x^k$, for some real or complex numbers $a_0, a_1, ...$

- Plug the power series expansions for y, p, q, and f into the DE (1).
- Comparing coeffiients of like powers of x, derive relations between the a_j's.
- Using these recurrance relations [R] and the ICs, solve for the coefficients of the power series of $y(x)$.

Example: Solve $y' - y = 5$, $y(0) = -4$, using the power series method.

This is easy to solve by undetermined coefficients: $y_h(x) = c_1e^x$ and $y_p(x) = A_1$. Solving for A_1 gives $A_1 = -5$ and then solving for c_1 gives $-4 = y(0) = -5 + c_1e^0$ so $c_1 = 1$ so $y = e^x - 5$.

Solving this using power series, we compute

[1] The diagrams were created using SAGE and and GIMP `http://www.gimp.org/` by the author. Originally written 9-26-2007. Some of the latex code is taken from the excellent (public domain!) text by Sean Mauch [M].

$$
\begin{array}{lll}
y'(x) & = & a_1 + 2a_2x + 3a_3x^2 + ... = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k \\
-y(x) & = & -a_0 - a_1x - a_2x^2 - ... = \sum_{k=0}^{\infty} -a_kx^k \\
---- & -- & -------------------------- \\
5 & = & (-a_0 + a_1) + (-a_1 + 2a_2)x + ... = \sum_{k=0}^{\infty}(-a_k + (k+1)a_{k+1})x^k
\end{array}
$$

Comparing coefficients,

- for $k = 0$: $5 = -a_0 + a_1$,
- for $k = 1$: $0 = -a_1 + 2a_2$,
- for general k: $0 = -a_k + (k+1)a_{k+1}$ for $k > 0$.

The IC gives us $-4 = y(0) = a_0$, so

$$a_0 = -4, \quad a_1 = 1, \quad a_2 = 1/2, \quad a_3 = 1/6, \quad \cdots \quad , a_k = 1/k!.$$

This implies

$$y(x) = -4 + x + x/2 + \ \cdots \ + x^k/k! + \ \cdots \ = -5 + e^x,$$

which is in agreement from the previous discussion.

Example: Solve Bessel's equation [B] of the 0-th order,

$$x^2y'' + xy' + x^2y = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

using the power series method.

This DE is so well-known (it has important applications to physics and engineering) that the series expansion has already been worked out (most texts on special functions or differential equations have this but an online reference is [B]). Its Taylor series expansion around 0 is:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!^2}\left(\frac{x}{2}\right)^{2m}$$

for all x. We shall see below that $y(x) = J_0(x)$.

Let us try solving this ourselves using the power series method. We compute

$$
\begin{array}{lcl}
x^2y''(x) & = & 0 + 0 \cdot x + 2a_2x^2 + 6a_3x^3 + 12a_4x^4 + ... = \sum_{k=0}^{\infty} k(k-1)a_kx^k \\
xy'(x) & = & 0 + a_1x + 2a_2x^2 + 3a_3x^3 + ... = \sum_{k=0}^{\infty} ka_kx^k \\
x^2y(x) & = & 0 + 0 \cdot x + a_0x^2 + a_1x^3 + ... = \sum_{k=2}^{\infty} a_{k-2}x^k \\
---- & -- & -------------------------- \\
0 & = & 0 + a_1x + (a_0 + 4a_2)x^2 + ... = a_1x + \sum_{k=2}^{\infty}(a_{k-2} + k^2a_k)x^k.
\end{array}
$$

By the ICs, $a_0 = 1$, $a_1 = 0$. Comparing coefficients,

$$k^2a_k = -a_{k-2}, \quad k \geq 2,$$

which implies

$$a_2 = -(\frac{1}{2})^2, \quad a_3 = 0, \quad a_4 = (\frac{1}{2} \cdot \frac{1}{4})^2, \quad a_5 = 0, \quad a_6 = -(\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{6})^2, \cdots .$$

In general,

$$a_{2k} = (-1)^k 2^{-2k} \frac{1}{k!^2}, \quad a_{2k+1} = 0,$$

for $k \geq 1$. This is in agreement with the series above for J_0.

Some of this computation can be formally done in SAGE using power series rings.

SAGE

```
sage: R6.<a0,a1,a2,a3,a4,a5,a6> = PolynomialRing(QQ,7)
sage: R.<x> = PowerSeriesRing(R6)
sage: y = a0 + a1*x + a2*x^2 + a3*x^3 + a4*x^4 + a5*x^5 +\
   a6*x^6 + O(x^7)
sage: y1 = y.derivative()
sage: y2 = y1.derivative()
sage: x^2*y2 + x*y1 + x^2*y
a1*x + (a0 + 4*a2)*x^2 + (a1 + 9*a3)*x^3 + (a2 + 16*a4)*x^4 +\
 (a3 + 25*a5)*x^5 + (a4 + 36*a6)*x^6 + O(x^7)
```

This is consistent with our "paper and pencil" computations above.

SAGE knows quite a few special functions, such as the various types of Bessel functions.

SAGE

```
sage: b = lambda x:bessel_J(x,0)
sage: P = plot(b,0,20,thickness=1)
sage: show(P)
sage: y = lambda x: 1 - (1/2)^2*x^2 + (1/8)^2*x^4 - (1/48)^2*x^6
sage: P1 = plot(y,0,4,thickness=1)
sage: P2 = plot(b,0,4,linestyle="--")
sage: show(P1+P2)
```

This is displayed below:

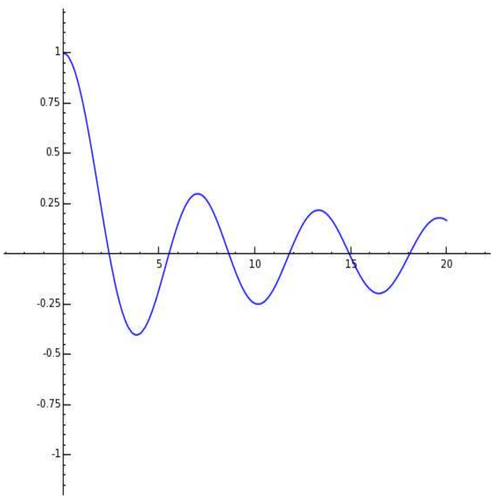

Figure 1: The Bessel function $J_0(x)$, for $0 < x < 20$.

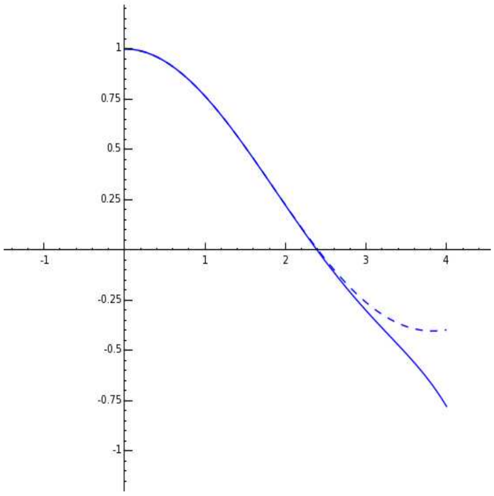

Figure 2: A Taylor polynomial approximation for $J_0(x)$.

Exercise: Use SAGE to find the first 5 terms in the power series solution to $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$. Plot this Taylor polynomial approximation over $-\pi < x < \pi$.

References

[B] Wikipedia entry for the Bessel functions:
`http://en.wikipedia.org/wiki/Bessel_function`

[P] Wikipedia entry for the power series method:
`http://en.wikipedia.org/wiki/Power_series_method`

[M] Sean Mauch, *Introduction to methods of Applied Mathematics*,
`http://www.its.caltech.edu/~sean/book/unabridged.html`

[R] Wikipedia entry for the recurrence relations:
`http://en.wikipedia.org/wiki/Recurrence_relations`