Newtonian mechanics - falling body problems

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We briefly recall how the physics of the falling body problem leads naturally to a differential equation (this was already mentioned in the introduction and forms a part of Newtonian mechanics [M]). Consider a mass m falling due to gravity. We orient coordinates to that downward is positive. Let $x(t)$ denote the distance the mass has fallen at time t and $v(t)$ its velocity at time t. We assume only two forces act: the force due to gravity, F_{grav}, and the force due to air resistence, F_{res}. In other words, we assume that the total force is given by

$$F_{total} = F_{grav} + F_{res}.$$

We know that $F_{grav} = mg$, where $g > 0$ is the gravitational constant, from high school physics. We assume, as is common in physics, that air resistance is proportional to velocity: $F_{res} = -kv = -kx'(t)$, where $k \geq 0$ is a constant. Newton's second law [N] tells us that $F_{total} = ma = mx''(t)$. Putting these all together gives $mx''(t) = mg - kx'(t)$, or

$$v'(t) + \frac{k}{m}v(t) = g. \tag{1}$$

This is the differential equation governing the motion of a falling body. Equation (1) can be solved by various methods: separation of variables or by integrating factors. If we assume $v(0) = v_0$ is given and if we assume $k > 0$ then the solution is

$$v(t) = \frac{mg}{k} + (v_0 - \frac{mg}{k})e^{-kt/m}. \tag{2}$$

In particular, we see that the limiting velocity is $v_{limit} = \frac{mg}{k}$.

Example 1. *Wile E. Coyote (see [W] if you haven't seen him before) has mass* 100 *kgs (with chute). The chute is released* 30 *seconds after the jump from a height of* 2000 *m. The force due to air resistence is given by* $\vec{F}_{res} = -k\vec{v}$*, where*

$$k = \begin{cases} 15, & \text{chute closed,} \\ 100, & \text{chute open.} \end{cases}$$

Find

(a) the distance and velocity functions during the time when the chute is closed (i.e., $0 \leq t \leq 30$ *seconds),*

(b) the distance and velocity functions during the time when the chute is open (i.e., $30 \leq t$ *seconds),*

(c) the time of landing,

(d) the velocity of landing. (Does Wile E. Coyote survive the impact?)

soln: *Taking* $m = 100$, $g = 9.8$, $k = 15$ *and* $v(0) = 0$ *in (2), we find*

$$v_1(t) = \frac{196}{3} - \frac{196}{3} e^{-\frac{3}{20}t}.$$

This is the velocity with the time t starting the moment the parachutist jumps. After $t = 30$ *seconds, this reaches the velocity* $v_0 = \frac{196}{3} - \frac{196}{3}e^{-9/2} = 64.607...$. *The distance fallen is*

$$\begin{array}{ll} x_1(t) & = \int_0^t v_1(u)\, du \\ & = \frac{196}{3}\, t + \frac{3920}{9}\, e^{-\frac{3}{20}t} - \frac{3920}{9}. \end{array}$$

After 30 *seconds, it has fallen* $x_1(30) = \frac{13720}{9} + \frac{3920}{9}e^{-9/2} = 1529.283...$ *meters.*

Taking $m = 100$, $g = 9.8$, $k = 100$ *and* $v(0) = v_0$*, we find*

$$v_2(t) = \frac{49}{5} + e^{-t}\left(\frac{833}{15} - \frac{196}{3}\, e^{-9/2}\right).$$

This is the velocity with the time t starting the moment Wile E. Coyote opens his chute (i.e., 30 *seconds after jumping). The distance fallen is*

$$\begin{array}{ll} x_2(t) & = \int_0^t v_2(u)\, du + x_1(30) \\ & = \frac{49}{5}\, t - \frac{833}{15}\, e^{-t} + \frac{196}{3}\, e^{-t}e^{-9/2} + \frac{71099}{45} + \frac{3332}{9}\, e^{-9/2}. \end{array}$$

Now let us solve this using SAGE *.*

SAGE
```
sage: RR = RealField(sci_not=0, prec=50, rnd='RNDU')
sage: t = var('t')
sage: v = function('v', t)
sage: m = 100; g = 98/10; k = 15
sage: de = lambda v: m*diff(v,t) + k*v - m*g
sage: desolve_laplace(de(v(t)),["t","v"],[0,0])
'196/3-196*%e^-(3*t/20)/3'
sage: soln1 = lambda t: 196/3-196*exp(-3*t/20)/3
sage: P1 = plot(soln1(t),0,30,plot_points=1000)
sage: RR(soln1(30))
64.607545559502
```

This solves for the velocity before the coyote's chute is opened, $0 < t < 30$*. The last number is the velocity Wile E. Coyote is traveling at the moment he opens his chute.*

SAGE
```
sage: t = var('t')
sage: v = function('v', t)
sage: m = 100; g = 98/10; k = 100
sage: de = lambda v: m*diff(v,t) + k*v - m*g
sage: desolve_laplace(de(v(t)),["t","v"],[0,RR(soln1(30))])
'631931*%e^-t/11530+49/5'
sage: soln2 = lambda t: 49/5+(631931/11530)*exp(-(t-30))
              + soln1(30) - (631931/11530) - 49/5
sage: RR(soln2(30))
64.607545559502
sage: RR(soln1(30))
64.607545559502
sage: P2 = plot(soln2(t),30,50,plot_points=1000)
sage: show(P1+P2)
```

This solves for the velocity after the coyote's chute is opened, $t > 30$*. The last command plots the velocity functions together as a single plot. (You would see* *a break in the graph if you omitted the* SAGE *'s* `plot` *option* `,plot_points=1000`*. That is because the number of samples taken of the function by default is not sufficient to capture the jump the function takes at* $t = 30$*.) The terms at the end of* `soln2` *were added to insure* $x_2(30) = x_1(30)$*.*

Next, we find the distance traveled at time t*:*

SAGE

```
age: integral(soln1(t),t)
3920*e^(-(3*t/20))/9 + 196*t/3
sage: x1 = lambda t: 3920*e^(-(3*t/20))/9 + 196*t/3
sage: RR(x1(30))
1964.8385851589
```

This solves for the distance the coyote traveled before the chute was open, $0 < t < 30$*. The last number says that he has gone about* 1965 *meters when he opens his chute.*

SAGE

```
sage: integral(soln2(t),t)
49*t/5 - (631931*e^(30 - t)/11530)
sage: x2 = lambda t: 49*t/5 - (631931*e^(30 - t)/11530)
           + x1(30) + (631931/11530) - 49*30/5
sage: RR(x2(30.7))
1999.2895090436
sage: P4 = plot(x2(t),30,50)
sage: show(P3+P4)
```

(Again, you see a break in the graph because of the round-off error.) The terms at the end of `x2` *were added to insure* $x_2(30) = x_1(30)$*. You know he is close to the ground at* $t = 30$*, and going quite fast (about 65 m/s!). It makes sense that he will hit the ground soon afterwards (with a large puff of smoke, if you've seen the cartoons), even though his chute will have slowed him down somewhat.*

The graph of the velocity $0 < t < 50$ *is in Figure 1. Notice how it drops at* $t = 30$ *when the chute is opened. The graph of the distance fallen* $0 < t < 50$ *is in Figure 2. Notice how it slows down at* $t = 30$ *when the chute is opened.*

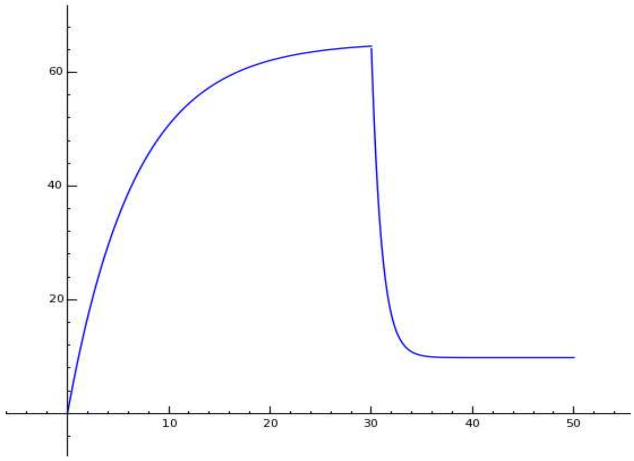

Figure 1: Velocity of falling parachutist.

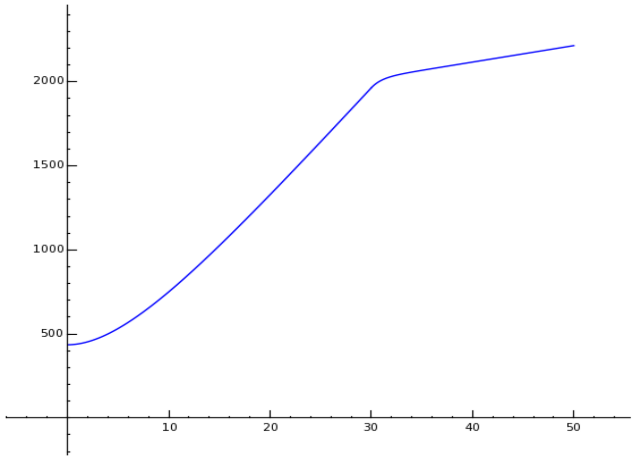

Figure 2: Distance fallen by a parachutist.

The time of impact is $t_{impact} = 30.7....$ *This was found numerically by a "trial-and-error" method of solving* $x_2(t) = 2000$.

The velocity of impact is $v_2(t_{impact}) \approx 37$ *m/s.*

Exercise: Drop an object with mass 10 kgs from a height of 2000 m. Suppose the force due to air resistence is given by $\vec{F}_{res} = -10\vec{v}$. Find the velocity after 10 seconds using SAGE . Plot this velocity function for $0 < t < 10$.

References

[BD] W. Boyce and R. DiPrima, **Elementary Differential Equations and Boundary Value Problems**, 8th edition, John Wiley and Sons, 2005.

[M] General wikipedia introduction to Newtonian mechanics
http://en.wikipedia.org/wiki/Classical_mechanics

[N] General wikipedia introduction to Newton's three laws of motion::
http://en.wikipedia.org/wiki/Newtons_Laws_of_Motion

[S] The SAGE Group, SAGE : *Mathematical software*, version 2.8.
http://www.sagemath.org/
http://sage.scipy.org/

[W] General wikipedia introduction to Wile E. Coyote and the RoadRunner:
http://en.wikipedia.org/wiki/Wile_E._Coyote_and_Road_Runner