Mixing problems in ODEs

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Suppose that we have two chemical substances where one is soluable in the other, such as salt and water. Suppose that we have a tank containing a mixture of these substances, and the mixture of them is poured in and the resulting "well-mixed" solution pours out through a value at the bottom. (The term "well-mixed" is used to indicate that the fluid being poured in is assumed to instantly dissolve into a homogeneous mixture the moment it goes into the tank.) The crude picture looks like this:

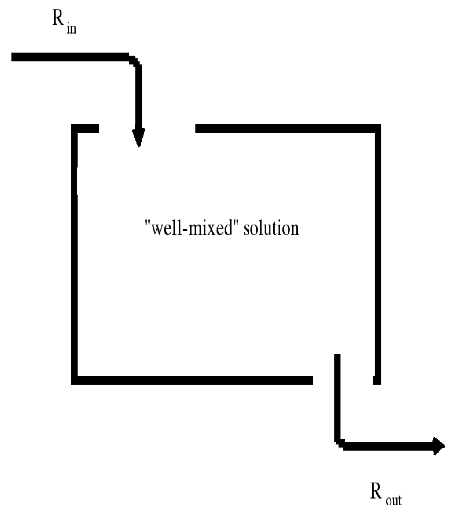

Figure 1: Solution pours into a tank, mixes with another type of solution. and then pours out.

Assume for concreteness that the chemical substances are salt and water. Let

- $A(t)$ denote the amount of salt at time t,
- FlowRateIn = the rate at which the solution pours into the tank,
- FlowRateOut = the rate at which the mixture pours out of the tank,
- C_{in} = "concentration in" = the concentration of salt in the solution being poured into the tank,
- C_{out} = "concentration out" = the concentration of salt in the solution being poured out of the tank,
- R_{in} = rate at which the salt is being poured into the tank = (FlowRateIn)(C_{in}),
- R_{out} = rate at which the salt is being poured out of the tank = (FlowRateOut)(C_{out}).

Remark 1. *Some things to make note of:*

- *If FlowRateIn = FlowRateOut then the "water level" of the tank stays the same.*
- *We can determine C_{out} as a function of other quantities:*

$$C_{out} = \frac{A(t)}{T(t)},$$

  where $T(t)$ denotes the volume of solution in the tank at time t.
- *The rate of change of the amount of salt in the tank, $A'(t)$, more properly could be called the "net rate of change". If you think if it this way then you see $A'(t) = R_{in} - R_{out}$.*

Now the differential equation for the amount of salt arises from the above equations:

$$A'(t) = (\text{FlowRateIn})C_{in} - (\text{FlowRateOut})\frac{A(t)}{T(t)}.$$

Example 1. *Consider a tank with 200 liters of salt-water solution, 30 grams of which is salt. Pouring into the tank is a brine solution at a rate of 4 liters/minute and with a concentration of 1 grams per liter. The "well-mixed" solution pours out at a rate of 5 liters/minute. Find the amount at time t.*

We know

$$A'(t) = (\text{FlowRateIn})C_{in} - (\text{FlowRateOut})\frac{A(t)}{T(t)} = 4 - 5\frac{A(t)}{200-t}, \quad A(0) = 30.$$

Writing this in the standard form $A' + pA = q$, we have

$$A(t) = \frac{\int \mu(t)q(t)\,dt + C}{\mu(t)},$$

where $\mu = e^{\int p(t)\,dt} = e^{-5\int \frac{1}{200-t}\,dt} = (200-t)^{-5}$ is the "integrating factor". This gives $A(t) = 200 - t + C\cdot(200-t)^5$, where the initial condition implies $C = -170\cdot 200^{-5}$.

Here is one way to do this using SAGE*:*

SAGE

```
sage: t = var('t')
sage: A = function('A', t)
sage: de = lambda A: diff(A,t) + (5/(200-t))*A - 4
sage: desolve(de(A(t)),[A,t])
'(%c-1/(t-200)^4)*(t-200)^5'
```

*This is the form of the general solution. (*SAGE *uses Maxima [M] and* `%c` *is Maxima's notation for an arbitrary constant.) Let us now solve this general solution for c, using the initial conditions.*

SAGE

```
sage: c = var('c')
sage: solnA = lambda t: (c - 1/(t-200)^4)*(t-200)^5
sage: solnA(t)
(c - (1/(t - 200)^4))*(t - 200)^5
sage: solnA(0)
-320000000000*(c - 1/1600000000)
```

```
sage: solve([solnA(0) == 30],c)
[c == 17/32000000000]
sage: c = 17/32000000000
sage: solnA(t)
(17/32000000000 - (1/(t - 200)^4))*(t - 200)^5
sage: P = plot(solnA(t),0,200)
sage: show(P)
```

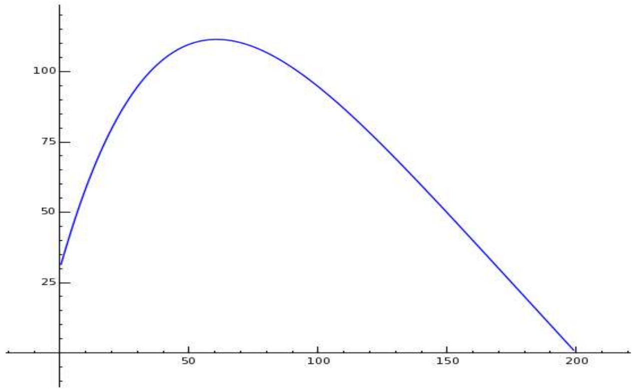

Figure 2: $A(t)$, $0 < t < 200$, $A' = 4 - 5A(t)/(200 - t)$, $A(0) = 30$.

Exercise: Now use SAGE to solve the same problem but with the same flow rate out as 4 liters/min (so the "water level" in the tank is constant). Find and plot the solution $A(t)$, $0 < t < 200$.

References

[BD] W. Boyce and R. DiPrima, **Elementary Differential Equations and Boundary Value Problems**, 8th edition, John Wiley and Sons, 2005.

[M] Maxima, a general purpose Computer Algebra system. `http://maxima.sourceforge.net/`