Linear ODEs, II

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To better describe the form a solution to a linear ODE can take, we need to better understand the nature of fundamental solutions and particular solutions.

Recall that the general solution to

$$y^{(n)} + b_1(t)y^{(n-1)} + ... + b_{n-1}(t)y' + b_n(t)y = f(t),$$

has the form $y = y_p + y_h$, where y_h is a linear combination of fundamental solutions. For example, the general solution to the spring-mass equation

$$x'' + x = 1$$

has the form $x = x(t) = 1 + c_1 \cos(t) + c_2 \sin(t)$ for the displacement from the equilibrium position. Suppose we are also given n initial conditions $y(x_0) = a_0$, $y'(x_0) = a_1$, $\ldots$, $y^{(n-1)}(x_0) = a_{n-1}$. For example, we could impose the initial position and initial velocity on the mass: $x(0) = x_0$ and $x'(0) = v_0$. Of course, no matter what x_0 and v_0 are are given, we want to be able to solve for the coefficients c_1, c_2 in $x(t) = 1 + c_1 \cos(t) + c_2 \sin(t)$ to obtain a unique solution. More generally, we want to be able to solve an n-th order IVP and obtain a unique solution. A few questions arise.

- How do we know this can be done?
- How do we know that there isn't a linear combination of fundamental solutions which isn't 0 (i.e., the zero function)?

The complete answer actually involves methods from linear algebra which go beyond this course. The basic idea though is not hard to understand and it involves what is called "the Wronskian[2]". We'll have to explain what this means first. If $f_1(t)$, $f_2(t)$, $\ldots$, $f_n(t)$ are given n-times differentiable functions then their **fundamental matrix** is the matrix

[2]Josef Wronski was a Polish-born French mathemtician who worked in many different areas of applied mathematics and mechanical engineering [Wr].

$$\Phi = \Phi(f_1, ..., f_n) = \begin{pmatrix} f_1(t) & f_2(t) & \dots & f_n(t) \\ f_1'(t) & f_2'(t) & \dots & f_n'(t) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \dots & f_n^{(n-1)}(t) \end{pmatrix}.$$

The determinant of the fundamental matrix is called the **Wronskian**, denoted $W(f_1, ..., f_n)$. The Wronskian actually helps us answer both questions above simultaneously.

Example 1 *Take* $f_1(t) = \sin^2(t)$, $f_2(t) = \cos^2(t)$, *and* $f_3(t) = 1$. SAGE *allows us to easily compute the Wronskian:*

SAGE

```
sage: SR = SymbolicExpressionRing()
sage: MS = MatrixSpace(SR,3,3)
sage: Phi = MS([[sin(t)^2,cos(t)^2,1],
                [diff(sin(t)^2,t),diff(cos(t)^2,t),0],
                [diff(sin(t)^2,t,t),diff(cos(t)^2,t,t),0]])
sage: Phi

[               sin(t)^2                cos(t)^2        1]
[       2*cos(t)*sin(t)       -2*cos(t)*sin(t)        0]
[2*cos(t)^2 - 2*sin(t)^2 2*sin(t)^2 - 2*cos(t)^2       0]
sage: Phi.det()
0
```

Here `Phi.det()` *is the determinant of the fundamental matrix* `Phi`. *Since it is zero, this means* $W(sin(t)^2, cos(t)^2, 1) = 0$. *(Note: the above entry for* `Phi` *should all be on one line. For typographical reasons, we have spread it out to 3 lines.) The entries for the symbolic expression ring* `SR` *and the* 3×3 *matrix space* `MS` *above are used to construct the matrix* `Phi` *having symbolic entries.*

We try one more example:

SAGE

```
sage: SR = SymbolicExpressionRing()
sage: MS = MatrixSpace(SR,2,2)
sage: Phi = MS([[sin(t)^2,cos(t)^2],
                [diff(sin(t)^2,t),diff(cos(t)^2,t)]])
sage: Phi

[        sin(t)^2          cos(t)^2]
[ 2*cos(t)*sin(t) -2*cos(t)*sin(t)]
sage: Phi.det()
-2*cos(t)*sin(t)^3 - 2*cos(t)^3*sin(t)
```

This means $W(sin(t)^2, cos(t)^2) = -2\cos(t)\sin(t)^3 - 2\cos(t)^3\sin(t)$*, which is non-zero.*

If there are constants c_1, ..., c_n, not all zero, for which

$$c_1 f_1(t) + c_2 f_2(t) \cdots + c_n f_n(t) = 0, \quad \text{for all } t, \tag{1}$$

then the functions f_i $(1 \leq i \leq n)$ are called **linearly dependent**. If the functions f_i $(1 \leq i \leq n)$ are not linearly dependent then they are called **linearly independent** (this definition is frequently seen for linearly independent vectors [L] but holds for functions as well). This condition (1) can be interpreted geometrically as follows. Just as $c_1x+c_2y = 0$ is a line through the origin in the plane and $c_1x+c_2y+c_3z = 0$ is a plane containing the origin in 3-space, the equation

$$c_1x_1 + c_2x_2 \cdots + c_nx_n = 0,$$

is a "hyperplane" containing the origin in n-space with coordinates $(x_1, ..., x_n)$. This condition (1) says geometrically that the graph of the space curve $\vec{r}(t) = (f_1(t), \ldots, f_n(t))$ lies entirely in this hyperplane. If you pick n functions "at random" then they are "probably" linearly independent, because "random" space curves don't lie in a hyperplane. But certainly not all collections of functions are linearly independent.

Example 2 *Consider just the two functions* $f_1(t) = \sin^2(t)$, $f_2(t) = \cos^2(t)$. *We know from the* SAGE *computation in the example above that these functions are linearly independent.*

SAGE

```
sage: P = parametric_plot((sin(t)^2,cos(t)^2),0,5)
sage: show(P)
```

The SAGE *plot of this space curve* $\vec{r}(t) = (\sin(t)^2, \cos(t)^2)$ *is given below. It is obviously not contained in a line through the origin, therefore making it geometrically clear that these functions are linearly independent.*

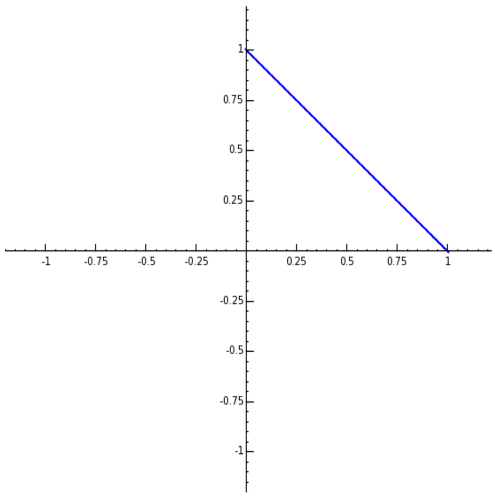

Figure 1: Parametric plot of $(\sin(t)^2, \cos(t)^2)$.

The following two results answer the above questions.

Theorem 3 *(***Wronskian test***) If* $f_1(t)$, $f_2(t)$, $\ldots$, $f_n(t)$ *are given n-times differentiable functions with a non-zero Wronskian then they are linearly independent.*

As a consequence of this theorem, and the SAGE computation in the example above, $f_1(t) = \sin^2(t)$, $f_2(t) = \cos^2(t)$, are linearly independent.

Theorem 4 *Given any homogeneous n-th linear ODE*

$$y^{(n)} + b_1(t)y^{(n-1)} + ... + b_{n-1}(t)y' + b_n(t)y = 0,$$

with differentiable coefficients, there always exists n solutions $y_1(t)$, ..., $y_n(t)$ which have a non-zero Wronskian.

The functions $y_1(t)$, ..., $y_n(t)$ in the above theorem are called **fundamental solutions**.

We shall not prove either of these theorems here. Please see [BD] for further details.

Exercise: Use SAGE to compute the Wronskian of
(a) $f_1(t) = \sin(t)$, $f_2(t) = \cos(t)$,
(b) $f_1(t) = 1$, $f_2(t) = t$, $f_3(t) = t^2$, $f_4(t) = t^3$.
Check that
(a) $y_1(t) = \sin(t)$, $y_2(t) = \cos(t)$ are fundamental solutions for $y'' + y = 0$,
(d) $y_1(t) = 1$, $y_2(t) = t$, $y_3(t) = t^2$, $y_4(t) = t^3$ are fundamental solutions for $y^{(4)} = y'''' = 0$.

References

[BD] W. Boyce and R. DiPrima, **Elementary Differential Equations and Boundary Value Problems**, 8th edition, John Wiley and Sons, 2005.

[L] General wikipedia introduction Linear Independence
`http://en.wikipedia.org/wiki/Linearly_independent`

[W] General wikipedia introduction to the Wronskian
`http://en.wikipedia.org/wiki/Wronskian`

[Wr] St. Andrews MacTutor entry for Wronski
`http://www-groups.dcs.st-and.ac.uk/%7Ehistory/Biographies/Wronski.html`