Initial value problems

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A **1-st order initial value problem**, or IVP, is simply a 1-st order ODE and an initial condition. For example,

$$x'(t) + p(t)x(t) = q(t), \qquad x(0) = x_0,$$

where $p(t)$, $q(t)$ and x_0 are given. The analog of this for 2nd order linear DEs is this:

$$a(t)x''(t) + b(t)x'(t) + c(t)x(t) = f(t), \qquad x(0) = x_0,\ x'(0) = v_0,$$

where $a(t)$, $b(t)$, $c(t)$, x_0, and v_0 are given. This 2-nd order linear DE and initial conditions is an example of a **2-nd order IVP**. In general, in an IVP, the number of initial conditions must match the order of the DE.

Example 1 *Consider the 2-nd order DE*

$$x'' + x = 0.$$

(We shall run across this DE many times later. As we will see, it represents the displacement of an undamped spring with a unit mass attached. The term **harmonic oscillator** *is attached to this situation [O].) Suppose we know that the general solution to this DE is*

$$x(t) = c_1 \cos(t) + c_2 \sin(t),$$

for any constants c_1, c_2. This means every solution to the DE must be of this form. (If you don't believe this, you can at least check it it is a solution by computing $x''(t)+x(t)$ and verifying that the terms cancel, as in the following SAGE *example. Later, we see how to derive this solution.) Note that there are two degrees of freedom (the constants c_1 and c_2), matching the order of the DE.*

SAGE

```
sage: t = var('t')
sage: c1 = var('c1')
sage: c2 = var('c2')
sage: de = lambda x: diff(x,t,t) + x
sage: de(c1*cos(t) + c2*sin(t))
0
sage: x = function('x', t)
sage: soln = desolve_laplace(de(x(t)),["t","x"],[0,0,1])
sage: soln
'sin(t)'
sage: solnx = lambda s: RR(eval(soln.replace("t","s")))
sage: P = plot(solnx,0,2*pi)
sage: show(P)
```

This is displayed below.

Now, to solve the IVP

$$x'' + x = 0, \quad x(0) = 0, \ x'(0) = 1.$$

the problem is to solve for c_1 *and* c_2 *for which the* $x(t)$ *satisfies the initial conditions. The two degrees of freedom in the general solution matching the number of initial conditions in the IVP. Plugging* $t = 0$ *into* $x(t)$ *and* $x'(t)$*, we obtain*

$$0 = x(0) = c_1 \cos(0) + c_2 \sin(0) = c_1, \quad 1 = x'(0) = -c_1 \sin(0) + c_2 \cos(0) = c_2.$$

Therefore, $c_1 = 0$*,* $c_2 = 1$ *and* $x(t) = \sin(t)$ *is the unique solution to the IVP.*

Here you see the solution oscillates, as t *gets larger.*

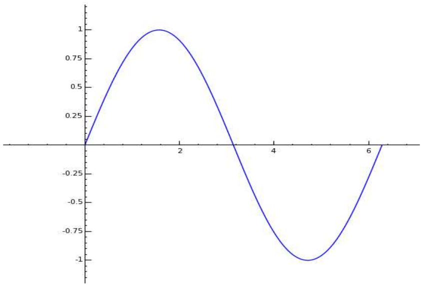

Figure 1: Solution to IVP $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$.

Another example,

Example 2 *Consider the 2-nd order DE*

$$x'' + 4x' + 4x = 0.$$

(We shall run across this DE many times later as well. As we will see, it represents the displacement of a critially damped spring with a unit mass attached.) Suppose we know that the general solution to this DE is

$$x(t) = c_1 exp(-2t) + c_2 t exp(-2t) = c_1 e^{-2t} + c_2 t e^{-2t},$$

for any constants c_1, c_2. This means every solution to the DE must be of this form. (Again, you can at least check it is a solution by computing $x''(t)$, $4x'(t)$, $4x(t)$, adding them up and verifying that the terms cancel, as in the following SAGE *example.)*

SAGE

```
sage: t = var('t')
sage: c1 = var('c1')
sage: c2 = var('c2')
sage: de = lambda x: diff(x,t,t) + 4*diff(x,t) + 4*x
sage: de(c1*exp(-2*t) + c2*t*exp(-2*t))
4*(c2*t*e^(-2*t) + c1*e^(-2*t)) + 4*(-2*c2*t*e^(-2*t)
+ c2*e^(-2*t) - 2*c1*e^(-2*t)) + 4*c2*t*e^(-2*t)
- 4*c2*e^(-2*t) + 4*c1*e^(-2*t)
sage: de(c1*exp(-2*t) + c2*t*exp(-2*t)).expand()
0
sage: desolve_laplace(de(x(t)),["t","x"],[0,0,1])
```

```
't*%e^-(2*t)'
sage: P = plot(t*exp(-2*t),0,pi)
sage: show(P)
```

The plot is displayed below.

Now, to solve the IVP

$$x'' + 4x' + 4x = 0, \quad x(0) = 0,\ x'(0) = 1.$$

we solve for c_1 *and* c_2 *using the initial conditions. Plugging* $t = 0$ *into* $x(t)$ *and* $x'(t)$*, we obtain*

$$0 = x(0) = c_1 \exp(0) + c_2 \cdot 0 \cdot \exp(0) = c_1,$$
$$1 = x'(0) = c_1 \exp(0) + c_2 \exp(0) - 2c_2 \cdot 0 \cdot \exp(0) = c_1 + c_2.$$

Therefore, $c_1 = 0$*,* $c_1 + c_2 = 1$ *and so* $x(t) = t \exp(-2t)$ *is the unique solution to the IVP. Here you see the solution tends to 0, as* t *gets larger.*

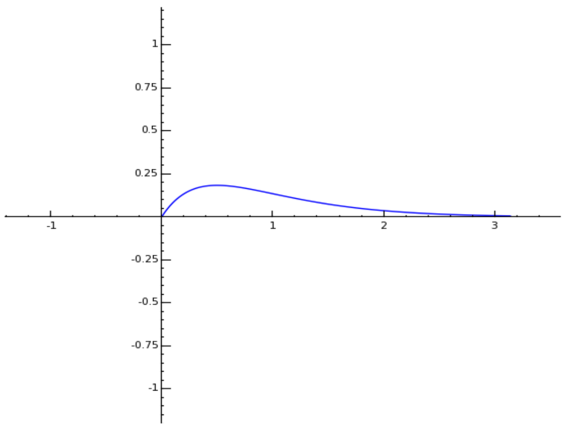

Figure 2: Solution to IVP $x'' + 4x' + 4x = 0$, $x(0) = 0$, $x'(0) = 1$.

Suppose, for the sake of argument, that I **lied** *to you and told you the general solution to this DE is*

$$x(t) = c_1 exp(-2t) + c_2 exp(-2t) = c_1(e^{-2t} + c_2 e^{-2t}),$$

for any constants c_1, c_2. (In other words, the "extra t factor" is missing.) Now, if you try to solve for the constant c_1 and c_2 using the initial conditions $x(0) = 0$, $x'(0) = 1$ you will get the equations

$$c_1 + c_2 = 0$$
$$-2c_1 - 2c_2 = 1.$$

These equations are impossible to solve! You see form this that you must have a correct general solution to insure that you can solve your IVP.

One more quick example.

Example 3 *Consider the 2-nd order DE*

$$x'' - x = 0.$$

Suppose we know that the general solution to this DE is

$$x(t) = c_1 exp(t) + c_2 exp(-t) = c_1 e^{-t} + c_2 e^{-t},$$

for any constants c_1, c_2. (Again, you can check it is a solution.)

The solution to the IVP

$$x'' - x = 0, \quad x(0) = 0, \ x'(0) = 1,$$

is $x(t) = \frac{e^t + e^{-t}}{2}$. (You can solve for c_1 and c_2 yourself, as in the examples above.) This particular function is also called a **hyperbolic cosine function**, *denoted* $\cosh(t)$.

The hyperbolic trig functions have many properties analogous to the usual trig functions and arise in many areas of applications[H]. For example, $\cosh(t)$ *represents a catenary or hanging cable [C].*

SAGE

```
sage: t = var('t')
sage: c1 = var('c1')
sage: c2 = var('c2')
sage: de = lambda x: diff(x,t,t) - x
sage: de(c1*exp(-t) + c2*exp(-t))
0
sage: desolve_laplace(de(x(t)),["t","x"],[0,0,1])
```

```
'%e^t/2-%e^-t/2'
sage: P = plot(e^t/2-e^(-t)/2,0,3)
sage: show(P)
```

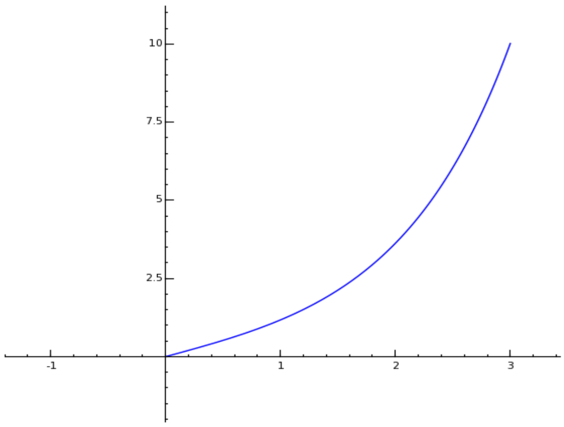

Figure 3: Solution to IVP $x'' - x = 0$, $x(0) = 0$, $x'(0) = 1$.

Here you see the solution tends to infinity, as t gets larger.

Exercise: The general solution to the falling body problem

$$mv' + kv = mg,$$

is $v(t) = \frac{mg}{k} + ce^{-kt/m}$. If $v(0) = v_0$, solve for c in terms of v_0. Take $m = k = v_0 = 1$, $g = 9.8$ and use SAGE to plot $v(t)$ for $0 < t < 1$.

References

[BD] W. Boyce and R. DiPrima, **Elementary Differential Equations and Boundary Value Problems**, 8th edition, John Wiley and Sons, 2005.

[C] General wikipedia introduction to the Catenary:
`http://en.wikipedia.org/wiki/Catenary`

[H] General wikipedia introduction to the Hyperbolic trig function
`http://en.wikipedia.org/wiki/Hyperbolic_function`

[O] General wikipedia introduction to the Harmonic oscillator
`http://en.wikipedia.org/wiki/Harmonic_oscillator`