First order ODEs - isoclines and direction fields

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Recall from vector calculus the notion of a two-dimensional vector field: $\vec{F}(x,y) = (g(x,y), h(x,y))$. To plot $\vec{F}$, you simply draw the vector $\vec{F}(x,y)$ at each point (x,y).

The idea of the **direction field** (or **slope field**) associated to the first order ODE

$$y' = f(x,y), \qquad y(a) = c, \tag{1}$$

is similar. At each point (x,y) you plot a vector having slope $f(x,y)$. For example, the vector field plot of $\vec{F}(x,y) = (1, f(x,y))$ or $\vec{F}(x,y) = (1, f(x,y))/\sqrt{1+f(x,y)^2}$ (which is a unit vector).

A related notion are the isoclines of the ODE. An **isocline** of (1) is a level curve of the function $z = f(x,y)$:

$$\{(x,y) \mid f(x,y) = m\},$$

where the given constant m is called the **slope** of the isocline. In terms of the ODE, this curve represents the collection of points at which the solution has slope m. In terms of the direction field of the ODE, it represents the collection of points where the vectors have slope m.

How to draw the direction field of (1) by hand:

- Draw several isoclines, making sure to include one which contains the point (a,c). (You may want to draw these in pencil.)
- On each isocline, draw "hatch marks" or "arrows" along the line each having slope m.

This is a crude direction field plot. The plot of arrows form your direction field. The isoclines, having served their usefulness, can safely be ignored or erased.

Example 1. *The direction field, with three isoclines, for*

$$y' = 5x + y - 5, \quad y(0) = 1,$$

is given by the following graph:

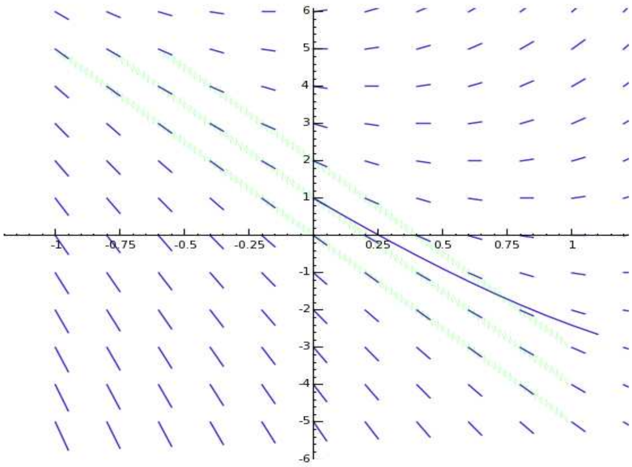

Figure 1: Plot of $y' = 5x + y - 5$, $y(0) = 1$, for $-1 < x < 1$.

The isoclines are the curves (coincidentally, lines) of the form $5x+y-5 = m$*. (They are green bands in the above plot.) These are lines of slope* -5*, not to be confused with the fact that it represents an isocline of slope* m*.*

The above example can be solved explicitly. (Indeed, $y = -5x + e^x$ solves $y' = 5x + y - 5$, $y(0) = 1$.) In the next example, such an explicit solution is (as far as I know), not possible. Therefore, a numerical approximation plays a more important role.

Example 2. *The direction field, with three isoclines, for*

$$y' = x^2 + y^2, \quad y(0) = 3/2,$$

is given by the following graph:

The isoclines are the concentric circles $x^2 + y^2 = m$*. (They are green in the above plot.)*

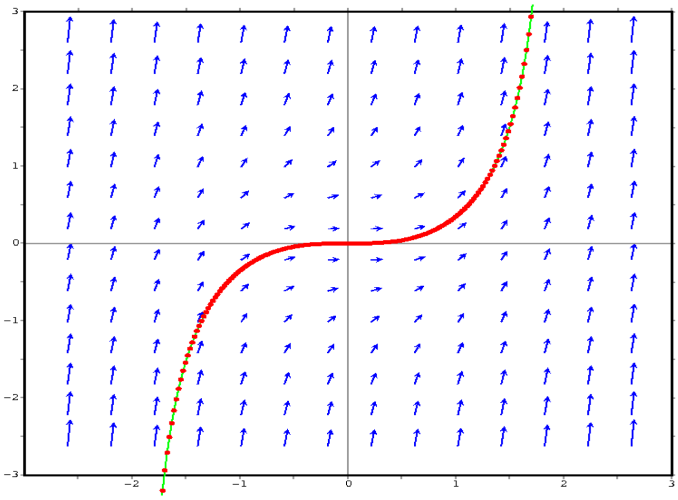

Figure 2: Direction field and solution plot of $y' = x^2 + y^2$, $y(0) = 3/2$, for $-3 < x < 3$.

The plot above was obtained using SAGE *'s interface with Maxima, and the plotting package Openmath (*SAGE *includes both Maxima and Openmath).* :

SAGE

```
sage: maxima.eval('load("plotdf")')
sage: maxima.eval('plotdf(x^2+y^2,[trajectory_at,0,0],
                                  [x,-3,3],[y,-3,3])')
```

This gave the above plot. (Note: the `plotdf` *command goes on one line; for typographical reasons, it was split in two.)*

There is also a way to draw these direction fields using SAGE .

SAGE

```
sage: pts = [(-2+i/5,-2+j/5) for i in range(20)
               for j in range(20)] # square [-2,2]x[-2,2]
sage: f = lambda p:p[0]^2+p[1]^2
sage: arrows = [arrow(p, (p[0]+0.02,p[1]+(0.02)*f(p)),
              width=1/100, rgbcolor=(0,0,1)) for p in pts]
sage: show(sum(arrows))
```

This gives the plot below.

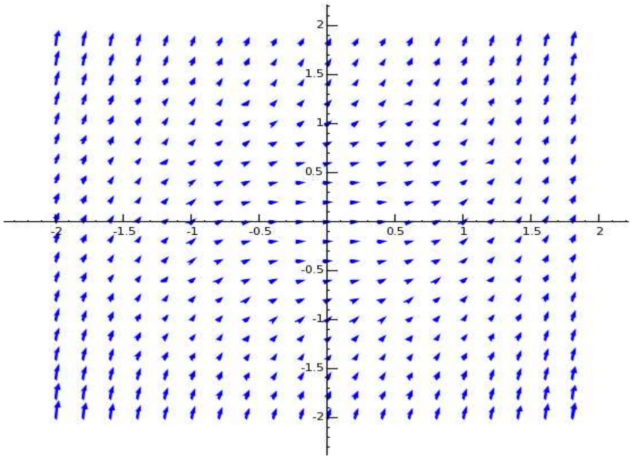

Figure 3: Direction field for $y' = x^2 + y^2$, $y(0) = 3/2$, for $-2 < x < 2$.

Exercise: Using SAGE, plot the direction field for $y' = x^2 - y^2$.

References

[BD] W. Boyce and R. DiPrima, **Elementary Differential Equations and Boundary Value Problems**, 8th edition, John Wiley and Sons, 2005.

[D] Wikipedia introduction to direction fields:
`http://en.wikipedia.org/wiki/Slope_field`

[DF] Direction Field Plotter of Prof Robert Israel:
`http://www.math.ubc.ca/~israel/applet/dfplotter/dfplotter.html`

[M] Maxima, a general purpose, open source computer algebra system:
`http://maxima.sourceforge.net/`