Introduction to DEs

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> But there is another reason for the high repute of mathematics: it is mathematics that offers the exact natural sciences a certain measure of security which, without mathematics, they could not attain.
>
> \- *Albert Einstein*

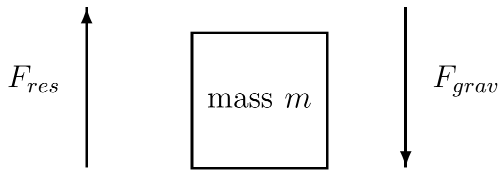
Motivation

Roughly speaking, a differential equation is an equation involving the derivatives of one or more unknown functions.

In calculus (differential, integral and vector), you've studied ways of analyzing functions. You might even have been convinced that functions you meet in applications arise naturally from physical principles. As we shall see, differential equations arise naturally from general physical principles. In many cases, the functions you met in calculus in applications to physics were actually solutions to a "natural" differential equation.

Example 1 *Consider a falling body of mass m on which exactly 3 forces act:*

- *gravitation, F_{grav},*
- *air resistance, F_{res},*
- *an external force, $F_{ext} = f(t)$, where $f(t)$ is some given function.*



Let $x(t)$ denote the distance fallen from some fixed initial position. The velocity is denoted by $v = x'$ and the acceleration by $a = x''$. We choose an orientation so that downwards is positive. In this case, $F_{grav} = mg$, where $g > 0$ is the gravitational constant. We assume that air resistance is proportional to velocity (a common assumption in physics), and write $F_{res} = -kv = -kx'$, where $k > 0$ is a "friction constant". The total force, F_{total}, is by hypothesis,

$$F_{total} = F_{grav} + F_{res} + F_{ext},$$

and, by Newton's 2nd Law[2]*,*

$$F_{total} = ma = mx''.$$

Putting these together, we have

$$mx'' = ma = mg - kx' + f(t),$$

or

$$mx'' + mx' = f(t) + mg.$$

This is a differential equation in $x = x(t)$. It may also be rewritten as a differential equation in $v = v(t) = x'(t)$ as

$$mv' + kv = f(t) + mg.$$

This is an example of a "first order differential equation in v", which means that at most first order derivatives of the unknown function $v = v(t)$ occur.

In fact, you have probably seen solutions to this in your calculus classes, at least when $f(t) = 0$ and $k = 0$. In that case, $v'(t) = g$ and so $v(t) = \int g\,dt = gt + C$. Here the constant of integration C represents the initial velocity.

[2] "Force equals mass times acceleration." `http://en.wikipedia.org/wiki/Newtons_law`

Differential equations occur in other areas as well: weather prediction (more generally, fluid-flow dynamics), electrical circuits, the heat of a homogeneous wire, and many others (see the table below). They even arise in problems on Wall Street: the Black-Scholes equation is a PDE which models the pricing of derivatives [BS]. Learning to solve differential equations helps understand the behaviour of phenomenon present in these problems.

phenomenon	description of DE
weather	Navier-Stokes equation [NS] a non-linear vector-valued higher-order PDE
falling body	1st order linear ODE
motion of a mass attached to a spring	Hooke's spring equation 2nd order linear ODE [H]
motion of a plucked guitar string	Wave equation 2nd order linear PDE [W]
Battle of Trafalger	Lanchester's equations system of 2 1st order DEs [L], [M], [N]
cooling cup of coffee in a room	Newton's Law of Cooling 1st order linear ODE
population growth	logistic equation non-linear, separable, 1st order ODE

Undefined terms and notation will be defined below, except for the equations themselves. For those, see the references or wait until later sections when they will be introduced[3].

Basic Concepts:

Here are some of the concepts to be introduced below:

- dependent variable(s),
- independent variable(s),
- ODEs,
- PDEs,

[3]Except for the Navier-Stokes equation, which is more complicated and might take us too far afield.

- order,
- linearity,
- solution.

It is really best to learn these concepts using examples. However, here are the general definitions anyway, with examples to follow.

The term "differential equation" is sometimes abbreviated DE, for brevity.

Dependent/independent variables: Put simply, a differential equation is an equation involving derivatives of one of more unknown functions. The variables you are differentiating with respect to are the **independent variables** of the DE. The variables (the "unknown functions") you are differentiating are the **dependent variables** of the DE. Other variables which might occur in the DE are sometimes called "parameters".

ODE/PDE: If none of the derivatives which occur in the DE are partial derivatives (for example, if the dependent variable/unknown function is a function of a single variable) then the DE is called an **ordinary differential equation** of **PDE**. If some of the derivatives which occur in the DE are partial derivatives then the DE is a **partial differential equation** or **PDE**.

Order: The highest total number of derivatives you have to take in the DE is it's **order**.

Linearity: This can be described in a few different ways. First of all, a DE is *linear* if the only operations you perform on its terms are combinations of the following:

- differentiation with respect to independent variable(s),
- multiplication by a function of the independent variable(s).

Another way to define linearity is as follows. A **linear ODE** having independent variable t and the dependent variable is y is an ODE of the form

$$a_0(t)y^{(n)} + ... + a_{n-1}(t)y' + a_n(t)y = f(t),$$

for some given functions $a_0(t)$, ..., $a_n(t)$, and $f(t)$. Here

$$y^{(n)} = y^{(n)}(t) = \frac{d^n y(t)}{dt^n}$$

denotes the n-th derivative of $y = y(t)$ with respect to t. The terms $a_0(t)$, $\ldots$, $a_n(t)$ are called the **coefficients** of the DE and we will call the term $f(t)$ the **non-homogeneous term** or the **forcing function**. (In physical applications, this term usually represents an external force acting on the system. For instance, in the example above it represents the gravitational force, mg.)

Solution: An explicit **solution** to a DE having independent variable t and the dependent variable is x is simple a function $x(t)$ for which the DE is true for all values of t.

Here are some examples:

Example 2 *Here is a table of examples. As an exercise, determine which of the following are ODEs and which are PDEs.*

DE	*indep vars*	*dep vars*	*order*	*linear?*
$mx'' + kx' = mg$ *falling body*	t	x	2	*yes*
$mv' + kv = mg$ *falling body*	t	v	1	*yes*
$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ *heat equation*	t, x	u	2	*yes*
$mx'' + bx' + kx = f(t)$ *spring equation*	t	x	2	*yes*
$P' = k(1 - \frac{P}{K})P$ *logistic population equation*	t	P	1	*no*
$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial^2 t}$ *wave equation*	t, x	u	2	*yes*
$T' = k(T - T_{room})$ *Newton's Law of Cooling*	t	T	1	*yes*
$x' = -Ay$, $y' = -Bx$, *Lanchester's equations*	t	x, y	1	*yes*

Remark: *Note that in many of these examples, the symbol used for the independent variable is not made explicit. For example, we are writing* x' *when we really mean* $x'(t) = \frac{x(t)}{dt}$. *This is very common shorthand notation and, in this situation, we shall usually use* t *as the independent variable whenever possible.*

Example 3 *Recall a linear ODE having independent variable t and the dependent variable is y is an ODE of the form*

$$a_0(t)y^{(n)} + ... + a_{n-1}(t)y' + a_n(t)y = f(t),$$

for some given functions $a_0(t)$, ..., $a_n(t)$, and $f(t)$. The order of this DE is n. In particular, a linear 1st order ODE having independent variable t and the dependent variable is y is an ODE of the form

$$a_0(t)y' + a_1(t)y = f(t),$$

for some $a_0(t)$, $a_1(t)$, and $f(t)$. We can divide both sides of this equation by the leading coefficient $a_0(t)$ without changing the solution y to this DE. Let's do that and rename the terms:

$$y' + p(t)y = q(t),$$

where $p(t) = a_1(t)/a_0(t)$ and $q(t) = f(t)/a_0(t)$. Every linear 1st order ODE can be put into this form, for some p and q. For example, the falling body equation $mv'+kv = f(t)+mg$ has this form after dividing by m and renaming v as y.

What does a differential equation like $mx''+kx' = mg$ or $P' = k(1-\frac{P}{K})P$ or $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial^2 t}$ really mean? In $mx'' + kx' = mg$, m and k and g are given constants. The only things that can vary are t and the unknown function $x = x(t)$.

Example 4 *To be specific, let's consider $x' + x = 1$. This means for all t, $x'(t) + x(t) = 1$. In other words, a solution $x(t)$ is a function which, when added to its derivative you always get the constant 1. How many functions are there with that property? Try guessing a few "random" functions:*

- *Guess $x(t) = \sin(t)$. Compute $(\sin(t))' + \sin(t) = \cos(t) + \sin(t) = \sqrt{2}\sin(t + \frac{\pi}{4})$. $x'(t) + x(t) = 1$ is false.*
- *Guess $x(t) = \exp(t) = e^t$. Compute $(e^t)' + e^t = 2e^t$. $x'(t) + x(t) = 1$ is false.*
- *Guess $x(t) = \exp(t) = t^2$. Compute $(t^2)' + t^2 = 2t + t^2$. $x'(t) + x(t) = 1$ is false.*

- *Guess* $x(t) = \exp(-t) = e^{-t}$. *Compute* $(e^{-t})' + e^{-t} = 0$. $x'(t) + x(t) = 1$ *is false.*

- *Guess* $x(t) = \exp(t) = 1$. *Compute* $(1)' + 1 = 0 + 1 = 1$. $x'(t) + x(t) = 1$ *is true.*

We finally found a solution by considering the constant function $x(t) = 1$. *Here a way of doing this kind of computation with the aid of the computer algebra system* SAGE *:*

SAGE

```
sage: t = var('t')
sage: de = lambda x: diff(x,t) + x - 1
sage: de(sin(t))
sin(t) + cos(t) - 1
sage: de(exp(t))
2*e^t - 1
sage: de(t^2)
t^2 + 2*t - 1
sage: de(exp(-t))
-1
sage: de(1)
0
```

Note we have rewritten $x' + x = 1$ *as* $x' + x - 1 = 0$ *and then plugged various functions for* x *to see if we get* 0 *or not.*

Obviously, we want a more systematic method for solving such equations than guessing all the types of functions we know one-by-one. We will get to those methods in time. First, we need some more terminology.

IVP: A first order **initial value problem** (abbreviated **IVP**) is a problem of the form

$$x' = f(t, x), \quad x(a) = c,$$

where $f(t, x)$ is a given function of two variables, and a, c are given constants. The equation $x(a) = c$ is the **initial condition**.

Under mild conditions of f, an IVP has a solution $x = x(t)$ which is unique. This means that if f and a are fixed but c is a parameter then the solution $x = x(t)$ will depend on c. This is stated more precisely in the following result.

Theorem 5 *(Existence and uniqueness) Fix a point (t_0, x_0) in the plane. Let $f(t, x)$ be a function of t and x for which both $f(t, x)$ and $f_x(t, x) = \frac{\partial f(t,x)}{\partial x}$ are continuous on some rectangle*

$$a < t < b, \quad c < x < d,$$

in the plane. Here a, b, c, d are any numbers for which $a < t_0 < b$ and $c < x_0 < d$. Then there is an $h > 0$ and a unique solution $x = x(t)$ for which

$$x' = f(t, x), \quad \text{for all } t \in (t_0 - h, t_0 + h),$$

and $x(t_0) = x_0$.

This is proven in §2.8 of Boyce and DiPrima [BD], but we shall not prove this here. In most cases we shall run across, it is easier to construct the solution than to prove this general theorem.

Example 6 *Let us try to solve*

$$x' + x = 1, \quad x(0) = 1.$$

The solutions to the DE $x' + x = 1$ which we "guessed at" in the previous example, $x(t) = 1$, satisfies this IVP.

Here a way of finding this slution with the aid of the computer algebra system SAGE *:*

SAGE

```
sage: t = var('t')
sage: x = function('x', t)
sage: de = lambda y: diff(y,t) + y - 1
sage: desolve_laplace(de(x(t)),["t","x"],[0,1])
'1'
```

(The command `desolve_laplace` *is a DE solver in* SAGE *which uses a special method involving Laplace transforms which we will learn later.) Just as an illustration, let's try another example. Let us try to solve*

$$x' + x = 1, \quad x(0) = 2.$$

The SAGE *commands are similar:*

SAGE

```
sage: t = var('t')
sage: x = function('x', t)
sage: de = lambda y: diff(y,t) + y - 1
sage: soln = desolve_laplace(de(x(t)),["t","x"],[0,2]); soln
'%e^-t+1'
sage: solnx = lambda s: RR(eval(soln.replace("^","**").
                 replace("%","").replace("t",str(s))))
sage: solnx(3)
1.04978706836786
sage: P = plot(solnx,0,5)
sage: show(P)
```

(The solnx line should all be typed on one line.) The plot is given below.

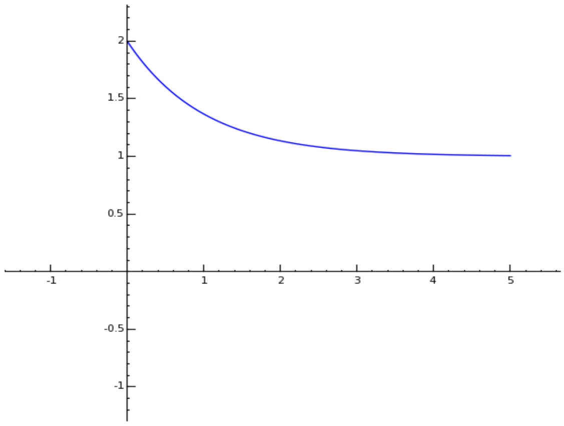


Figure 1: Solution to IVP $x' + x = 1$, $x(0) = 2$.

Exercise: Verify the, for any constant c, the function $x(t) = 1 + ce^{-t}$ solves $x' + x = 1$. Find the c for which this function solves the IVP $x' + x = 1$, $x(0) = 3$.. Solve this (a) by hand, (b) using SAGE .

References

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[W] General wikipedia introduction to the Wave equation: http://en.wikipedia.org/wiki/Wave_equation