The heat equation

Prof. Joyner[1]

> The deep study of nature is the most fruitful source of mathematical discoveries.
>
> - *Jean-Baptist-Joseph Fourier*

The heat equation with *zero ends* boundary conditions models the temperature of an (insulated) wire of length L:

$$\begin{cases} k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \\ u(0,t) = u(L,t) = 0. \end{cases}$$

Here $u(x,t)$ denotes the temperature at a point x on the wire at time t. The initial temperature $f(x)$ is specified by the equation

$$u(x,0) = f(x).$$

Method:

- Find the sine series of $f(x)$:

$$f(x) \sim \sum_{n=1}^{\infty} b_n(f) \sin(\frac{n\pi x}{L}),$$

- The solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(f) \sin(\frac{n\pi x}{L}) \exp(-k(\frac{n\pi}{L})^2 t).$$

Example: Let

$$f(x) = \begin{cases} -1, & 0 \leq x \leq \pi/2, \\ 2, & \pi/2 < x < \pi. \end{cases}$$

Then $L = \pi$ and

$$b_n(f) = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx = -2\,\frac{2\,\cos(n\pi) - 3\,\cos(\frac{1}{2}\,n\pi) + 1}{n\pi}.$$

Thus

$$f(x) \sim b_1(f)\sin(x) + b_2(f)\sin(2x) + ... = \frac{2}{\pi}\sin(x) - \frac{6}{\pi}\sin(2x) + \frac{2}{3\pi}\sin(3x) + ...$$

This can also be done in SAGE:

SAGE

```
sage: f1 = lambda x: -1
sage: f2 = lambda x: 2
sage: f = Piecewise([[(0,pi/2),f1],[(pi/2,pi),f2]])
sage: P1 = f.plot()
sage: b10 = [f.sine_series_coefficient(n,pi) for n in range(1,10)]
sage: b10
[2/pi, -6/pi, 2/(3*pi), 0, 2/(5*pi), -2/pi, 2/(7*pi), 0, 2/(9*pi)]
sage: ss10 = sum([b10[n]*sin((n+1)*x) for n in range(len(b50))])
sage: ss10
2*sin(9*x)/(9*pi) + 2*sin(7*x)/(7*pi) - 2*sin(6*x)/pi
+ 2*sin(5*x)/(5*pi) + 2*sin(3*x)/(3*pi) - 6*sin(2*x)/pi + 2*sin(x)/pi
sage: b50 = [f.sine_series_coefficient(n,pi) for n in range(1,50)]
sage: ss50 = sum([b50[n]*sin((n+1)*x) for n in range(len(b))])
sage: P2 = ss10.plot(-5,5,linestyle="--")
sage: P3 = ss50.plot(-5,5,linestyle=":")
sage: (P1+P2+P3).show()
```

This illustrates how the series converges to the function. The function $f(x)$, and some of the partial sums of its sine series, looks like Figure 1.

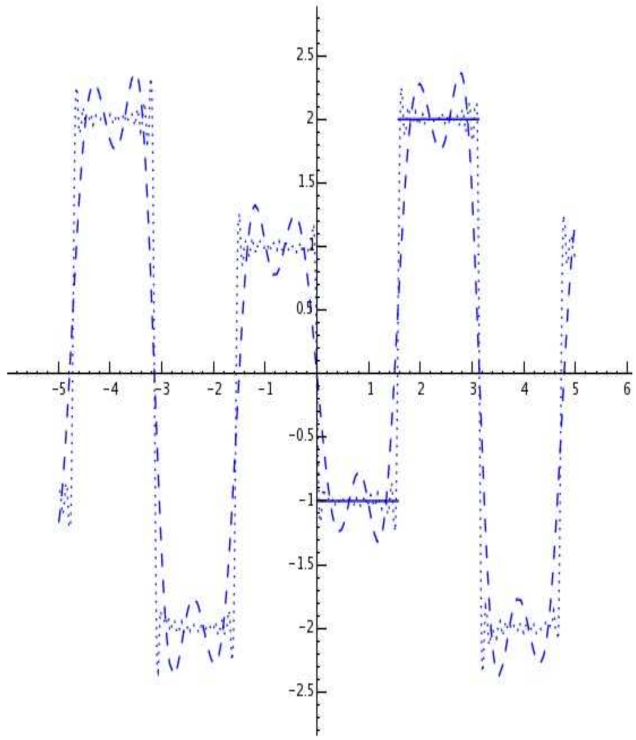

Figure 1: $f(x)$ and two sine series approximations.

As you can see, taking more and more terms gives functions which better and better approximate $f(x)$.

The solution to the heat equation, therefore, is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(f)\sin(\frac{n\pi x}{L})\exp(-k(\frac{n\pi}{L})^2 t).$$

Next, we see how SAGE can plot the solution to the heat equation (we use $k = 1$):

SAGE

```
sage: t = var("t")
sage: soln50 = sum([b[n]*sin((n+1)*x)*e^(-(n+1)^2*t) for n in range(len(b50))])
sage: soln50a = sum([b[n]*sin((n+1)*x)*e^(-(n+1)^2*(1/10)) for n in range(len(b50))])
sage: P4 = soln50a.plot(0,pi,linestyle=":")
sage: soln50b = sum([b[n]*sin((n+1)*x)*e^(-(n+1)^2*(1/2)) for n in range(len(b50))])
sage: P5 = soln50b.plot(0,pi)
sage: soln50c = sum([b[n]*sin((n+1)*x)*e^(-(n+1)^2*(1/1)) for n in range(len(b50))])
sage: P6 = soln50c.plot(0,pi,linestyle="--")
```

```
sage: (P1+P4+P5+P6).show()
```

Taking 50 terms of this series, the graph of the solution at $t = 0$, $t = 0.5$, $t = 1$, looks approximately like Figure 2.

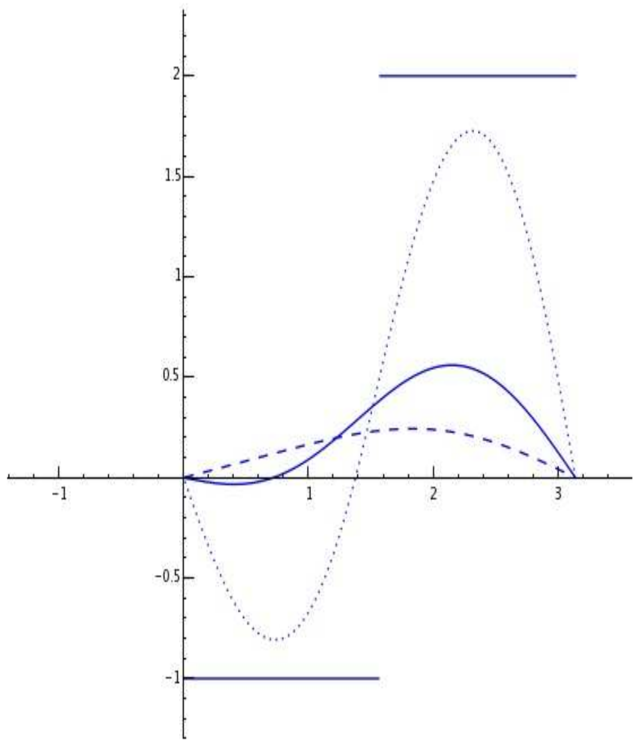

Figure 2: $f(x)$, $u(x, 0.1)$, $u(x, 0.5)$, $u(x, 1.0)$ using 60 terms of the sine series.

The heat equation with *insulated ends* boundary conditions models the temperature of an (insulated) wire of length L:

$$\begin{cases} k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \\ u_x(0,t) = u_x(L,t) = 0. \end{cases}$$

Here $u_x(x,t)$ denotes the partial derivative of the temperature at a point x on the wire at time t. The initial temperature $f(x)$ is specified by the equation $u(x,0) = f(x)$.

Method:

- Find the cosine series of $f(x)$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(\frac{n\pi x}{L}),$$

- The solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(\frac{n\pi x}{L}))\exp(-k(\frac{n\pi}{L})^2 t).$$

Example:
Let

$$f(x) = \begin{cases} -1, & 0 \le x \le \pi/2, \\ 2, & \pi/2 < x < \pi. \end{cases}$$

Then $L = \pi$ and

$$a_n(f) = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx = -6\frac{\sin\left(\frac{1}{2}\pi n\right)}{\pi n},$$

for $n > 0$ and $a_0 = 1$.

Thus

$$f(x) \sim \frac{a_0}{2} + a_1(f)\cos(x) + a_2(f)\cos(2x) + ...$$

This can also be done in SAGE:

SAGE

```
sage: f1 = lambda x: -1
sage: f2 = lambda x: 2
sage: f = Piecewise([[(0,pi/2),f1],[(pi/2,pi),f2]])
sage: P1 = f.plot()
sage: a10 = [f.cosine_series_coefficient(n,pi) for n in range(10)]
sage: a10
[1, -6/pi, 0, 2/pi, 0, -6/(5*pi), 0, 6/(7*pi), 0, -2/(3*pi)]
sage: a50 = [f.cosine_series_coefficient(n,pi) for n in range(50)]
sage: cs10 = a10[0]/2 + sum([a10[n]*cos(n*x) for n in range(1,len(a10))])
sage: P2 = cs10.plot(-5,5,linestyle="--")
```

```
sage: cs50 = a50[0]/2 + sum([a50[n]*cos(n*x) for n in range(1,len(a50))])
sage: P3 = cs50.plot(-5,5,linestyle=":")
sage: (P1+P2+P3).show()
```

This illustrates how the series converges to the function. The piecewise constant function $f(x)$, and some of the partial sums of its cosine series (one using 10 terms and one using 50 terms), looks like Figure 3.

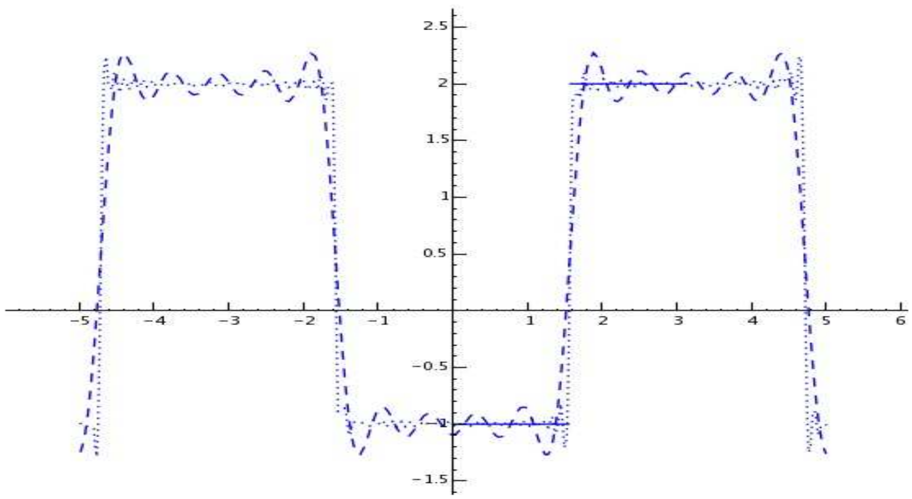

Figure 3: $f(x)$ and two cosine series approximations.

As you can see, taking more and more terms gives functions which better and better approximate $f(x)$.

The solution to the heat equation, therefore, is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(f)\cos(\frac{n\pi x}{L})\exp(-k(\frac{n\pi}{L})^2 t).$$

Using SAGE, we can plot this function:

SAGE

```
sage: soln50a = a50[0]/2 + sum([a50[n]*cos(n*x)*e^(-(n+1)^2*(1/100)) for n in range(1,len(a50))])
sage: soln50b = a50[0]/2 + sum([a50[n]*cos(n*x)*e^(-(n+1)^2*(1/10)) for n in range(1,len(a50))])
sage: soln50c = a50[0]/2 + sum([a50[n]*cos(n*x)*e^(-(n+1)^2*(1/2)) for n in range(1,len(a50))])
sage: P4 = soln50a.plot(0,pi)
sage: P5 = soln50b.plot(0,pi,linestyle=":")
```

```
sage: P6 = soln50c.plot(0,pi,linestyle="--")
sage: (P1+P4+P5+P6).show()
```

Taking only the first 50 terms of this series, the graph of the solution at $t = 0$, $t = 0.01$, $t = 0.1$,, $t = 0.5$, looks approximately like:

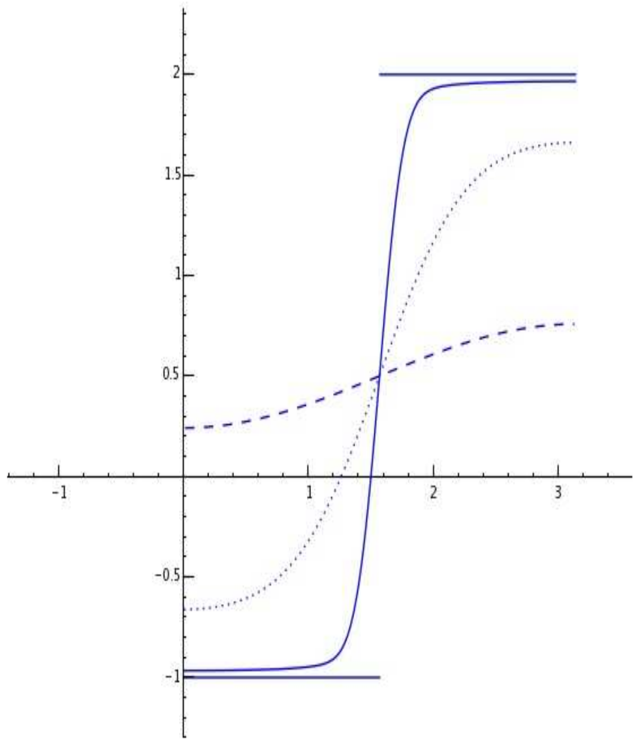

Figure 4: $f(x) = u(x, 0)$, $u(x, 0.01)$, $u(x, 0.1)$, $u(x, 0.5)$ using 50 terms of the cosine series.

Explanation:

Where does this solution come from? It comes from the method of separation of variables and the superposition principle. Here is a short explanation. We shall only discuss the "zero ends" case (the "insulated ends" case is similar).

First, assume the solution to the PDE $k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$ has the "factored" form

$$u(x,t) = X(x)T(t),$$

for some (unknown) functions X, T. If this function solves the PDE then it must satisfy $kX''(x)T(t) = X(x)T'(t)$, or

$$\frac{X''(x)}{X(x)} = \frac{1}{k}\frac{T'(t)}{T(t)}.$$

Since x, t are independent variables, these quotients must be constant. In other words, there must be a constant C such that

$$\frac{T'(t)}{T(t)} = kC, \quad X''(x) - CX(x) = 0.$$

Now we have reduced the problem of solving the one PDE to two ODEs (which is good), but with the price that we have introduced a constant which we don't know, namely C (which maybe isn't so good). The first ODE is easy to solve:

$$T(t) = A_1 e^{kCt},$$

for some constant A_1. To obtain physically meaningful solutions, we do not want the temperature of the wire to become unbounded as time increased (otherwise, the wire would simply melt eventually). Therefore, we may assume here that $C \leq 0$. It is best to analyse two cases now:

Case $C = 0$: This implies $X(x) = A_2 + A_3 x$, for some constants A_2, A_3. Therefore

$$u(x,t) = A_1(A_2 + A_3 x) = \frac{a_0}{2} + b_0 x,$$

where (for reasons explained later) $A_1 A_2$ has been renamed $\frac{a_0}{2}$ and $A_1 A_3$ has been renamed b_0.

Case $C < 0$: Write (for convenience) $C = -r^2$, for some $r > 0$. The ODE for X implies $X(x) = A_2 \cos(rx) + A_3 \sin(rx)$, for some constants A_2, A_3. Therefore

$$u(x,t) = A_1 e^{-kr^2t}(A_2\cos(rx) + A_3\sin(rx)) = (a\cos(rx) + b\sin(rx))e^{-kr^2t},$$

where A_1A_2 has been renamed a and A_1A_3 has been renamed b.

These are the solutions of the heat equation which can be written in factored form. By superposition, "the general solution" is a sum of these:

$$\begin{aligned} u(x,t) &= \tfrac{a_0}{2} + b_0 x + \textstyle\sum_{n=1}^{\infty}(a_n\cos(r_n x) + b_n\sin(r_n x))e^{-kr_n^2t} \\ &= \tfrac{a_0}{2} + b_0 x + (a_1\cos(r_1 x) + b_1\sin(r_1 x))e^{-kr_1^2t} \\ &\quad +(a_2\cos(r_2 x) + b_2\sin(r_2 x))e^{-kr_2^2t} + ..., \end{aligned} \qquad (1)$$

for some a_i, b_i, r_i. We may order the r_i's to be strictly increasing if we like.

We have not yet used the IC $u(x,0) = f(x)$ or the BCs $u(0,t) = u(L,t) = 0$. We do that next.

What do the BCs tell us? Plugging in $x = 0$ into (1) gives

$$0 = u(0,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-kr_n^2t} = \frac{a_0}{2} + a_1 e^{-kr_1^2t} + a_2 e^{-kr_2^2t} + ... \,.$$

These exponential functions are linearly independent, so $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, ... This implies

$$u(x,t) = b_0 x + \sum_{n=1} b_n \sin(r_n x)e^{-kr_n^2t} = b_0 x + b_1\sin(r_1 x)e^{-kr_1^2t} + b_2\sin(r_2 x)e^{-kr_2^2t} + ... \,.$$

Plugging in $x = L$ into this gives

$$0 = u(L,t) = b_0 L + \sum_{n=1} b_n \sin(r_n L)e^{-kr_n^2t}.$$

Again, exponential functions are linearly independent, so $b_0 = 0$, $b_n \sin(r_n L)$ for $n = 1, 2, ...$ In other to get a non-trivial solution to the PDE, we don't want $b_n = 0$, so $\sin(r_n L) = 0$. This forces $r_n L$ to be a multiple of π, say $r_n = n\pi/L$. This gives

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x) e^{-k(\frac{n\pi}{L})^2 t} = b_1 \sin(\frac{\pi}{L}x)) e^{-k(\frac{\pi}{L})^2 t} + b_2 \sin(\frac{2\pi}{L}x)) e^{-k(\frac{2\pi}{L})^2 t} + ..., \tag{2}$$

for some b_i's. The special case $t = 0$ is the so-called "sine series" expansion of the initial temperature function $u(x, 0)$. This was discovered by Fourier. To solve the heat eqution, it remains to solve for the "sine series coefficients" b_i.

There is one remaining condition which our solution $u(x,t)$ must satisfy. What does the IC tell us? Plugging $t = 0$ into (2) gives

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x) = b_1 \sin(\frac{\pi}{L}x)) + b_2 \sin(\frac{2\pi}{L}x)) + ... \ .$$

In other words, if $f(x)$ is given as a sum of these sine functions, or if we can somehow express $f(x)$ as a sum of sine functions, then we can solve the heat equation. In fact there is a formula[2] for these coefficients b_n:

$$b_n = \frac{2}{L} \int_0^L f(x) \cos(\frac{n\pi}{L}x) dx.$$

It is this formula which is used in the solutions above.

References

[BD] W. Boyce and R. DiPrima, **Elementary Differential Equations and Boundary Value Problems**, 8th edition, John Wiley and Sons, 2005.

[F1] Wikipedia biography of Fourier:
`http://en.wikipedia.org/wiki/Joseph_Fourier`

[F2] MacTutor biography of Fourier:
`http://www-groups.dcs.st-and.ac.uk/%7Ehistory/Biographies/Fourier.html`

[H] General wikipedia introduction to the heat equation:
`http://en.wikipedia.org/wiki/Heat_equation`

[2]Fourier did not know this formula at the time; it was discovered later by Dirichlet.

[S] The SAGE Group, SAGE : *Mathematical software*, version 2.8.
http://www.sagemath.org/
http://sage.scipy.org/