

Fourier series, sine series, cosine series

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History: Fourier series were discovered by J. Fourier, a Frenchman who was a mathematician among other things. In fact, Fourier was Napoleon's scientific advisor during France's invasion of Egypt in the late 1800's. When Napoleon returned to France, he "elected" (i.e., appointed) Fourier to be a Prefect - basically an important administrative post where he oversaw some large construction projects, such as highway constructions. It was during this time when Fourier worked on the theory of heat on the side. His solution to the heat equation is basically what undergraduates often learn in a DEs with BVPs class. The exception being that our understanding of Fourier series now is much better than what was known in the early 1800's and some of these facts, like Dirichlet's theorem, are covered as well.

Motivation: Fourier series, sine series, and cosine series are all expansions for a function $f(x)$, much in the same way that a Taylor series $a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$ is an expansion. Both Fourier and Taylor series can be used to approximate $f(x)$. There are at least three important differences between the two types of series. (1) For a function to have a Taylor series it must be differentiable², whereas for a Fourier series it does not even have to be continuous. (2) Another difference is that the Taylor series is typically not periodic (though it can be in some cases), whereas a Fourier series is *always* periodic. (3) Finally, the Taylor series (when it converges) always converges to the function $f(x)$, but the Fourier series may not (see Dirichlet's theorem below for a more precise description of what happens).

Definitions: Let $f(x)$ be a function defined on an interval of the real line. We allow $f(x)$ to be discontinuous but the points in this interval where $f(x)$ is discontinuous must be finite in number and must be jump discontinuities.

First, we discuss Fourier series. To have a Fourier series you must be given two things: (1) a "period" $P = 2L$, (2) a function $f(x)$ defined on an interval

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²Remember the formula for the n -th Taylor series coefficient centered at $x = x_0$ - $a_n = \frac{f^{(n)}(x_0)}{n!}$?

of length $2L$, usually we take $-L < x < L$ (but sometimes $0 < x < 2L$ is used instead). The **Fourier series of $f(x)$ with period $2L$** is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

where a_n and b_n are given by the formulas³,

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad (1)$$

and

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \quad (2)$$

Next, we discuss cosine series. To have a cosine series you must be given two things: (1) a “period” $P = 2L$, (2) a function $f(x)$ defined on the interval of length L , $0 < x < L$. The **cosine series of $f(x)$ with period $2L$** is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where a_n is given by

$$a_n = \frac{2}{L} \int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x) dx.$$

The cosine series of $f(x)$ is exactly the same as the Fourier series of the **even extension** of $f(x)$, defined by

$$f_{\text{even}}(x) = \begin{cases} f(x), & 0 < x < L, \\ f(-x), & -L < x < 0. \end{cases}$$

Finally, we define sine series. To have a sine series you must be given two things: (1) a “period” $P = 2L$, (2) a function $f(x)$ defined on the interval of length L , $0 < x < L$. The **sine series of $f(x)$ with period $2L$** is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

³These formulas were not known to Fourier. To compute the Fourier coefficients a_n, b_n he used sometimes ingenious round-about methods using large systems of equations.

where b_n is given by

$$b_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x) dx.$$

The sine series of $f(x)$ is exactly the same as the Fourier series of the **odd extension** of $f(x)$, defined by

$$f_{\text{odd}}(x) = \begin{cases} f(x), & 0 < x < L, \\ -f(-x), & -L < x < 0. \end{cases}$$

One last definition: the symbol \sim is used above instead of $=$ because of the fact that was pointed out above: the Fourier series may not converge to $f(x)$. Do you remember right-hand and left-hand limits from calculus 1? Recall they are denoted $f(x+) = \lim_{\epsilon \rightarrow 0, \epsilon > 0} f(x+\epsilon)$ and $f(x-) = \lim_{\epsilon \rightarrow 0, \epsilon > 0} f(x-\epsilon)$, resp.. The meaning of \sim is that the series does necessarily not converge to the value of $f(x)$ at every point⁴. The convergence properties are given by the theorem below.

Dirichlet's theorem⁵: Let $f(x)$ be a function as above and let $-L < x < L$. The Fourier series of $f(x)$,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

(where a_n and b_n are as in the formulas (1), (2)) converges to

$$\frac{f(x+) + f(x-)}{2}.$$

In other words, the Fourier series of $f(x)$ converges to $f(x)$ only if $f(x)$ is continuous at x . If $f(x)$ is not continuous at x then the Fourier series of $f(x)$ converges to the “midpoint of the jump”.

Examples: (1) If $f(x) = 2 + x$, $-2 < x < 2$, then the definition of L implies $L = 2$. Without even computing the Fourier series, we can evaluate it using Dirichlet's theorem.

Question: Using periodicity and Dirichlet's theorem, find the value that the Fourier series of $f(x)$ converges to at $x = 1, 2, 3$. (Ans: $f(x)$ is continuous at 1, so the Fourier series at $x = 1$ converges to $f(1) = 3$ by Dirichlet's theorem.)

⁴Fourier believed his series converged to the function in the early 1800's but we now know this is not always true.

⁵Pronounced “Dear-ish-lay”.

$f(x)$ is not defined at 2. Its Fourier series is periodic with period 4, so at $x = 2$ the Fourier series converges to $\frac{f(2+)+f(2-)}{2} = \frac{0+4}{2} = 2$. $f(x)$ is not defined at 3. Its Fourier series is periodic with period 4, so at $x = 3$ the Fourier series converges to $\frac{f(-1)+f(-1+)}{2} = \frac{1+1}{2} = 1$.)

The formulas (1) and (2) enable us to compute the Fourier series coefficients a_0 , a_n and b_n . (We skip the details.) These formulas give that the Fourier series of $f(x)$ is

$$f(x) \sim \frac{4}{2} + \sum_{n=1}^{\infty} -4 \frac{n\pi \cos(n\pi)}{n^2\pi^2} \sin\left(\frac{n\pi x}{2}\right).$$

The Fourier series approximations to $f(x)$ are

$$S_0 = 2, \quad S_1 = 2 + \frac{4}{\pi} \sin\left(\frac{\pi x}{2}\right), \quad S_2 = 2 + 4 \frac{\sin\left(\frac{1}{2} \pi x\right)}{\pi} - 2 \frac{\sin(\pi x)}{\pi}, \quad \dots$$

The graphs of each of these functions get closer and closer to the graph of $f(x)$ on the interval $-2 < x < 2$. For instance, the graph of $f(x)$ and of S_8 are given below:

Notice that $f(x)$ is only defined from $-2 < x < 2$ yet the Fourier series is not only defined everywhere but is periodic with period $P = 2L = 4$. Also, notice that S_8 is not a bad approximation to $f(x)$.

This can also be done in **SAGE**. First, we define the function.

```

_____ SAGE _____
sage: f = lambda x:x+2
sage: f = Piecewise([[-2,2),f]])

```

This can be plotted using the command `f.plot().show()`. Next, we compute the Fourier series coefficients:

```

_____ SAGE _____
sage: f.fourier_series_cosine_coefficient(0,2) # a_0
4
sage: f.fourier_series_cosine_coefficient(1,2) # a_1
0
sage: f.fourier_series_cosine_coefficient(2,2) # a_2

```

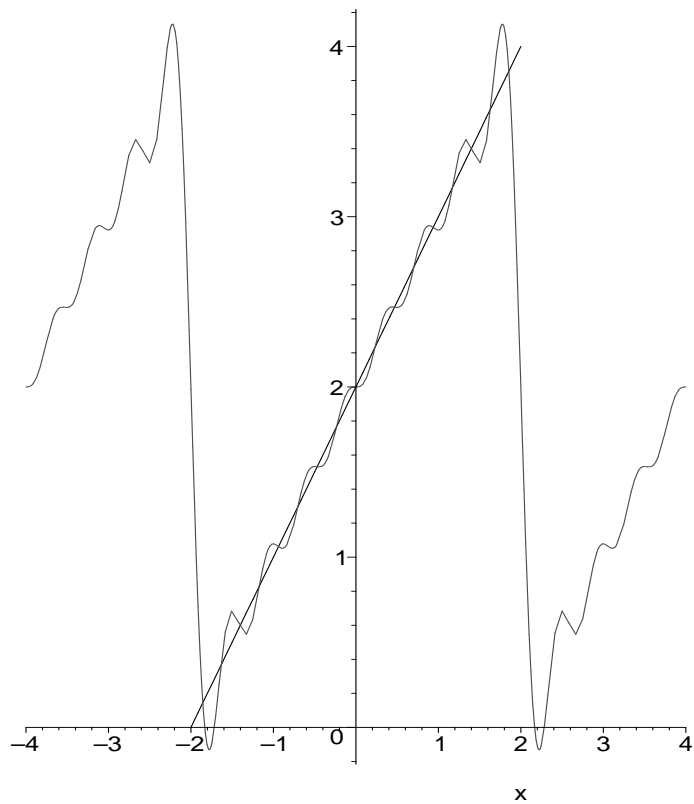


Figure 1: Graph of $f(x)$ and a Fourier series approximation of $f(x)$.

```

0
sage: f.fourier_series_cosine_coefficient(3,3) # a_3
0
sage: f.fourier_series_sine_coefficient(1,2) # b_1
4/pi
sage: f.fourier_series_sine_coefficient(2,2) # b_2
-2/pi
sage: f.fourier_series_sine_coefficient(3,2) # b_3
4/(3*pi)

```

Finally, the partial Fourier series and its plot versus the function can be computed using the following **SAGE** commands.

```

SAGE
sage: f.fourier_series_partial_sum(3,2)

```

```

-2*sin(pi*x)/pi + 4*sin(pi*x/2)/pi + 2
sage: P1 = f.plot_fourier_series_partial_sum(15,2,-5,5,linestyle=":")
sage: P2 = f.plot(rgbcolor=(1,1/4,1/2))
sage: (P1+P2).show()

```

The plot (which takes 15 terms of the Fourier series) is given below.

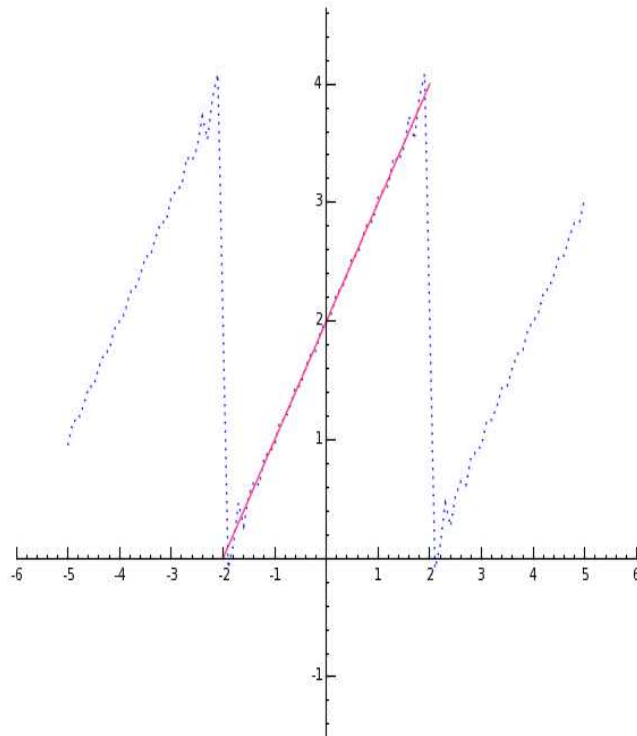


Figure 2: Graph of $f(x) = x + 2$ and a Fourier series approximation, $L = 2$.

(2) This time, let's consider an example of a cosine series. In this case, we take the piecewise constant function $f(x)$ defined on $0 < x < 3$ by

$$f(x) = \begin{cases} 1, & 0 < x < 2, \\ -1, & 2 \leq x < 3. \end{cases}$$

We see therefore $L = 3$. The formula above for the cosine series coefficients gives that

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} 4 \frac{\sin\left(\frac{2}{3} n\pi\right)}{n\pi} \cos\left(\frac{n\pi x}{3}\right).$$

The first few partial sums are

$$S_2 = 1/3 + 2 \frac{\sqrt{3} \cos\left(\frac{1}{3} \pi x\right)}{\pi},$$

$$S_3 = 1/3 + 2 \frac{\sqrt{3} \cos\left(\frac{1}{3} \pi x\right)}{\pi} - \frac{\sqrt{3} \cos\left(\frac{2}{3} \pi x\right)}{\pi}, \dots$$

As before, the more terms in the cosine series we take, the better the approximation is, for $0 < x < 3$. Comparing the picture below with the picture above, note that even with more terms, this approximation is not as good as the previous example. The precise reason for this is rather technical but basically boils down to the following: roughly speaking, the more differentiable the function is, the faster the Fourier series converges (and therefore the better the partial sums of the Fourier series will approximate $f(x)$). Also, notice that the cosine series approximation S_{10} is an even function but $f(x)$ is not (it's only defined from $0 < x < 3$).

For instance, the graph of $f(x)$ and of S_{10} are given below:

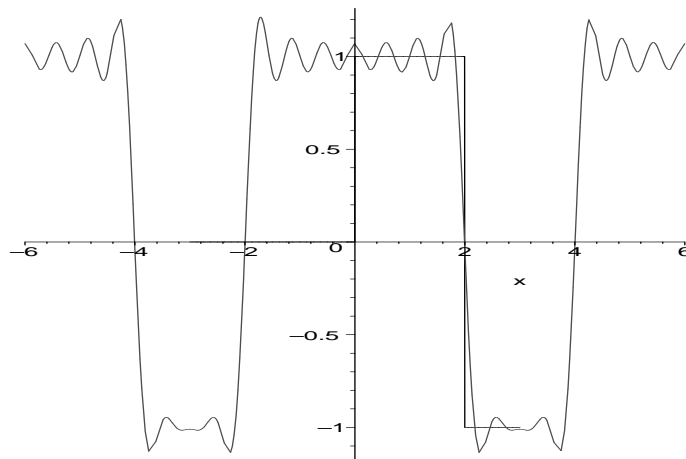


Figure 3: Graph of $f(x)$ and a cosine series approximation of $f(x)$.

(3) Finally, let's consider an example of a sine series. In this case, we

take the piecewise constant function $f(x)$ defined on $0 < x < 3$ by the same expression we used in the cosine series example above.

Question: Using periodicity and Dirichlet's theorem, find the value that the sine series of $f(x)$ converges to at $x = 1, 2, 3$. (Ans: $f(x)$ is continuous at 1, so the Fourier series at $x = 1$ converges to $f(1) = 1$. $f(x)$ is not continuous at 2, so at $x = 2$ the Sine series converges to $\frac{f(2+) + f(2-)}{2} = \frac{f(-2+) + f(2-)}{2} = \frac{-1+1}{2} = 0$. $f(x)$ is not defined at 3. It's Sine series is periodic with period 6, so at $x = 3$ the Sine series converges to $\frac{f_{\text{odd}}(3-) + f_{\text{odd}}(3+)}{2} = \frac{-1+1}{2} = 0$.)

The formula above for the sine series coefficients give that

$$f(x) = \sum_{n=1}^{\infty} 2 \frac{\cos(n\pi) - 2 \cos\left(\frac{2}{3}n\pi\right) + 1}{n\pi} \sin\left(\frac{n\pi x}{3}\right).$$

The partial sums are

$$S_2 = 2 \frac{\sin(1/3 \pi x)}{\pi} + 3 \frac{\sin\left(\frac{2}{3} \pi x\right)}{\pi},$$

$$S_3 = 2 \frac{\sin\left(\frac{1}{3} \pi x\right)}{\pi} + 3 \frac{\sin\left(\frac{2}{3} \pi x\right)}{\pi} - 4/3 \frac{\sin(\pi x)}{\pi}, \dots$$

These partial sums S_n , as $n \rightarrow \infty$, converge to their limit about as fast as those in the previous example. Instead of taking only 10 terms, this time we take 40. Observe from the graph below that the value of the sine series at $x = 2$ does seem to be approaching 0, as Dirichlet's Theorem predicts. The graph of $f(x)$ with S_{40} is

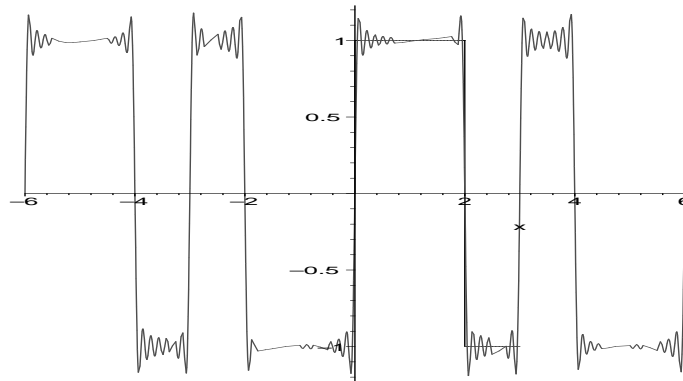


Figure 4: Graph of $f(x)$ and a sine series approximation of $f(x)$.

Exercise: Let $f(x) = x^2$, $-2 < x < 2$ and $L = 2$. Use **SAGE** to compute the first 10 terms of the Fourier series, and plot the corresponding partial sum. Next plot the partial sum of the first 50 terms and compare them.

References

- [F1] Wikipedia Fourier series article
http://en.wikipedia.org/wiki/Fourier_series

- [F2] MacTutor Fourier biography:
<http://www-groups.dcs.st-and.ac.uk/%7Ehistory/Biographies/Fourier.html>