Existence of solutions to ODEs

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[Peano] was a man I greatly admired from the moment I met him for the first time in 1900 at a Congress of Philosophy, which he dominated by the exactness of his mind.

-Bertrand Russell, 1932

When do solutions to an ODE exist? When are they unique? This section gives some necessary conditions for determining existence and uniqueness.

1 First order ODEs

We begin by considering the first order initial value problem

$$x'(t) = f(t, x(t)), \quad x(a) = c. \tag{1}$$

What conditions on f (and a and c) guarantee that a solution $x = x(t)$ exists? If it exists, what (further) conditions guarantee that $x = x(t)$ is unique?

The following result addresses the first question.

Theorem 1 *("Peano's existence theorem") Suppose f is bounded and continuous in x, and t. Then, for some value $\epsilon > 0$, there exists a solution $x = x(t)$ to the initial value problem within the range $[a - \epsilon, a + \epsilon]$.*

Giuseppe Peano (18581932) was an Italian mathematician, who is mostly known for his important work on the logical foundations of mathematics. For example, the common notations for union $\cup$ and intersections $\cap$ first appeared in his first book dealing with mathematical logic, written while he was teaching at the University of Turin.

 Much of this section uses Wikipedia [P], [PL].

Example 2 *Take* $f(x,t) = x^{2/3}$. *This is continuous and bounded in* x *and* t *in* $-1 < x < 1$, $t \in \mathbb{R}$. *The IVP* $x' = f(x,t)$, $x(0) = 0$ *has* **two** *solutions,* $x(t) = 0$ *and* $x(t) = t^3/27$.

You all know what continuity means but you may not be familiar with the slightly stronger notion of "Lipschitz continuity". This is defined next.

Definition 3 *Let* $D \subset \mathbb{R}^2$ *be a domain. A function* $f : D \to \mathbb{R}$ *is called Lipschitz continuous if there exists a real constant* $K > 0$*such that, for all* $x_1, x_2 \in D$,

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|.$$

The smallest such K *is called the Lipschitz constant of the function* f *on* D.

For example,

- the function $f(x) = x^{2/3}$ defined on $[-1, 1]$ is not Lipschitz continuous;
- the function $f(x) = x^2$ defined on $[-3, 7]$ is Lipschitz continuous, with Lipschitz constant $K = 14$;
- the function f defined by $f(x) = x^{3/2}\sin(1/x)$ $(x \neq 0)$ and $f(0) = 0$ restricted to $[0, 1]$, gives an example of a function that is differentiable on a compact set while not being Lipschitz.

Theorem 4 *("Picard's existence and uniqueness theorem") Suppose* f *is bounded, Lipschitz continuous in* x, *and continuous in* t. *Then, for some value* $\epsilon > 0$, *there exists a unique solution* $x = x(t)$ *to the initial value problem (1) within the range* $[a - \epsilon, a + \epsilon]$.

Charles Émile Picard (1856-1941) was a leading French mathematician. Picard made his most important contributions in the fields of analysis, function theory, differential equations, and analytic geometry. In 1885 Picard was appointed to the mathematics faculty at the Sorbonne in Paris. Picard was awarded the Poncelet Prize in 1886, the Grand Prix des Sciences Mathmatiques in 1888, the Grande Croix de la Légion d'Honneur in 1932, the Mittag-Leffler Gold Medal in 1937, and was made President of the International Congress of Mathematicians in 1920. He is the author of many books and his collected papers run to four volumes.

The proofs of Peano's theorem or Picard's theorem go *well* beyond the scope of this course. However, for the curious, a very brief indication of the main ideas will be given in the sketch below. For details, see an advanced text on differential equations.

sketch or the idea of the proof: A simple proof of existence of the solution is obtained by successive approximations. In this context, the method is known as Picard iteration.

Set $x_0(t) = c$ and

$$x_i(t) = c + \int_a^t f(s, x_{i-1}(s))\, ds.$$

It turns out that Lipschitz continuity implies that the mapping T defined by

$$T(y)(t) = c + \int_a^t f(s, y(s))\, ds,$$

is a contraction mapping on a certain Banach space. It can then be shown, by using the Banach fixed point theorem, that the sequence of "Picard iterates" x_i is convergent and that the limit is a solution to the problem. The proof of uniqueness uses a result called Grönwall's Lemma. $\Box$

Example 5 *Consider the IVP*

$$x' = 1 - x, \qquad x(0) = 1,$$

with the constant solution $x(t) = 1$*. Computing the Picard iterates by hand is easy:* $x_0(t) = 1$*,* $x_1(t) = 1 + \int_0^t 1 - x_0(s))\, ds = 1$*,* $x_2(t) = 1 + \int_0^t 1 - x_1(s))\, ds = 1$*, and so on. Since each* $x_i(t) = 1$*, we find the solution*

$$x(t) = \lim_{i \to \infty} x_i(t) = \lim_{i \to \infty} 1 = 1.$$

We now try the Picard iteration method in SAGE. Consider the IVP

$$x' = 1 - x, \qquad x(0) = 2,$$

which we considered earlier.

SAGE

```
sage: var('t, s')
sage: f = lambda t,x: 1-x
sage: a = 0; c = 2
sage: x0 = lambda t: c; x0(t)
2
sage: x1 = lambda t: c + integral(f(s,x0(s)), s, a, t); x1(t)
2 - t
sage: x2 = lambda t: c + integral(f(s,x1(s)), s, a, t); x2(t)
t^2/2 - t + 2
sage: x3 = lambda t: c + integral(f(s,x2(s)), s, a, t); x3(t)
-t^3/6 + t^2/2 - t + 2
sage: x4 = lambda t: c + integral(f(s,x3(s)), s, a, t); x4(t)
t^4/24 - t^3/6 + t^2/2 - t + 2
sage: x5 = lambda t: c + integral(f(s,x4(s)), s, a, t); x5(t)
-t^5/120 + t^4/24 - t^3/6 + t^2/2 - t + 2
sage: x6 = lambda t: c + integral(f(s,x5(s)), s, a, t); x6(t)
t^6/720 - t^5/120 + t^4/24 - t^3/6 + t^2/2 - t + 2
sage: P1 = plot(x2(t), t, 0, 2, linestyle='--')
sage: P2 = plot(x4(t), t, 0, 2, linestyle='-.')
sage: P3 = plot(x6(t), t, 0, 2, linestyle=':')
sage: P4 = plot(1+exp(-t), t, 0, 2)
sage: (P1+P2+P3+P4).show()
```

From the graph you can see how well these iterates are (or at least appear to be) converging to the true solution $x(t) = 1 + e^{-t}$.

More generally, here is some SAGE code for Picard iteration:

SAGE

```
def picard_iteration(f, a, c, N):
    '''
    Computes the N-th Picard iterate for the IVP

        x' = f(t,x), x(a) = c.

    EXAMPLES:
        sage: var('x t s')
        (x, t, s)
        sage: a = 0; c = 2
        sage: f = lambda t,x: 1-x
        sage: picard_iteration(f, a, c, 0)
```

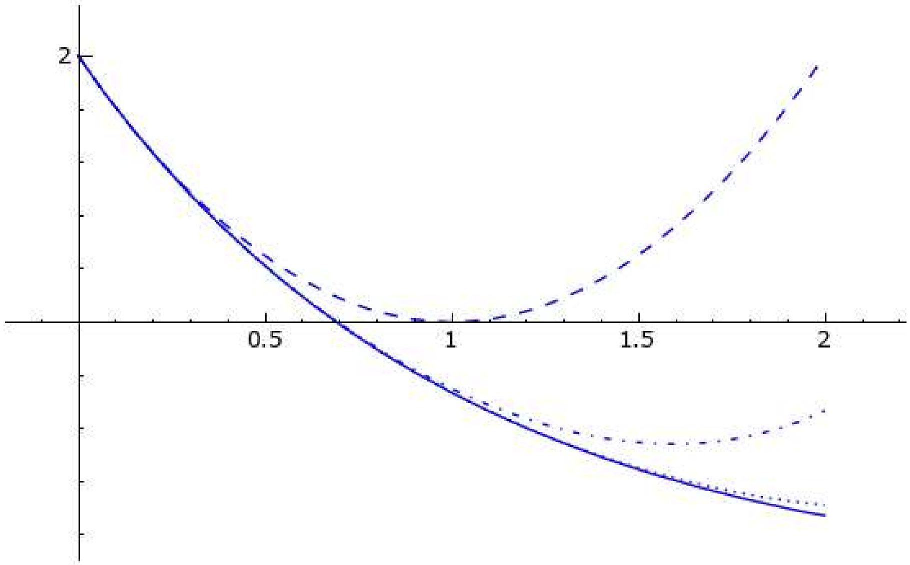

Figure 1: Picard iteration for $x' = 1 - x$, $x(0) = 2$.

```
    2
     sage: picard_iteration(f, a, c, 1)
    2 - t
    sage: picard_iteration(f, a, c, 2)
    t^2/2 - t + 2
    sage: picard_iteration(f, a, c, 3)
    -t^3/6 + t^2/2 - t + 2

'''
if N == 0:
    return c*t**0
if N == 1:
    #print integral(f(s,c*s**0), s, a, t)
    x0 = lambda t: c + integral(f(s,c*s**0), s, a, t)
    return expand(x0(t))
for i in range(N):
    x_old = lambda s: picard_iteration(f, a, c, N-1).subs(t=s)
    #print x_old(s)
    x0 = lambda t: c + integral(f(s,x_old(s)), s, a, t)
return expand(x0(t))
```

Exercise: Apply the Picard iteration method in SAGE to the IVP

$$x' = (t+x)^2, \qquad x(0) = 2,$$

and find the first three iterates.

2 Higher order constant coefficient linear homogeneous ODEs

We begin by considering the second order[2] initial value problem

$$ax'' + bx' + cx = 0, \quad x(0) = d_0, \quad x'(0) = d_1, \tag{2}$$

where a, b, c, d_0, d_1 are constants and $a \neq 0$. What conditions guarantee that a solution $x = x(t)$ exists? If it exists, what (further) conditions guarantee that $x = x(t)$ is unique? It turns out that no conditions are needed - a solution to 2 always exists and is unique. As we will see later, we can construct distinct explicit solutions, denoted $x_1 = x_1(t)$ and $x_2 = x_2(t)$ and sometimes called **fundamental solutions**, to $ax'' + bx' + cx = 0$. If we let $x = c_1x_1 + c_2x_2$, for any constants c_1 and c_2, then we know that x is also a solution[3], sometimes called the **general solution** to $ax'' + bx' + cx = 0$. But how do we know there exist c_1 and c_2 for which this general solution also satisfies the initial conditions $x(0) = d_0$ and $x'(0) = d_1$? For this to hold, we need to be able to solve

$$c_1x_1(0) + c_2x_2(0) = d_1, \qquad c_1x_1'(0) + c_2x_2'(0) = d_2,$$

for c_1 and c_2. By Cramer's rule,

$$c_1 = \frac{\begin{vmatrix} d_1 & x_2(0) \\ d_2 & x_2'(0) \end{vmatrix}}{\begin{vmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} x_1(0) & d_1 \\ x_1'(0) & d_2 \end{vmatrix}}{\begin{vmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{vmatrix}}.$$

For this solution to exist, the denominators in these quotients must be non-zero. This denominator is the value of the "Wronskian" at $t = 0$.

[2] It turns out that the reasoning in the second order case is very similar to the general reasoning for n-th order DEs. For simplicity of presentation, we restrict to the 2-nd order case.

[3] This follows form the linearity assumption.

Definition 6 *For* n *functions* $f_1, \ldots, f_n$, *which are* $n-1$ *times differentiable on an interval* I, *the* **Wronskian** $W(f_1, ..., f_n)$ *as a function on* I *is defined by*

$$W(f_1, \ldots, f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix},$$

for $x \in I$.

The matrix constructed by placing the functions in the first row, the first derivative of each function in the second row, and so on through the $(n-1)$-st derivative, is a square matrix sometimes called a **fundamental matrix** of the functions. The Wronskian is the determinant of the fundamental matrix.

Theorem 7 *("Abel's identity") Consider a homogeneous linear second-order ordinary differential equation*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\, y = 0$$

on the real line with a continuous function p. *The Wronskian* W *of two solutions of the differential equation satisfies the relation*

$$W(x) = W(0) \exp\left(-\int_0^x P(s)\, ds\right).$$

Definition 8 *We say* n *functions* $f_1, \ldots, f_n$ *are* **linearly dependent** *over the interval I, if there are numbers* $a_1, \cdots, a_n$ *(not all of them zero) such that*

$$a_1 f_1(x) + \cdots + a_n f_n(x) = 0,$$

for $x \in I$. *If the functions are not linearly dependent then they are called* **linearly independent**.

Theorem 9 *If the Wronskian is non-zero at some point in an interval, then the associated functions are linearly independent on the interval.*

Example 10 *If* $f_1(t) = e^t$ *and* $f_2(t) = e^{-t}$ *then*

$$\begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2.$$

Indeed,

SAGE

```
sage: var('t')
t
sage: f1 = exp(t); f2 = exp(-t)
sage: wronskian(f1,f2)
-2
```

Therefore, the fundamental solutions $x_1 = e^t$, $x_2 = e^{-t}$ *are*

Exercise: Using SAGE , verify Abel's identity in this example.

References

[P] General wikipedia introduction to the Peano existence theorem: http://en.wikipedia.org/wiki/Peano_existence_theorem

[PL] General wikipedia introduction to the Picard existence theorem: http://en.wikipedia.org/wiki/Picard-Lindelof_theorem

[PL] General wikipedia introduction to the Wronskian: http://en.wikipedia.org/wiki/Wronskian