Eigenvalue method for systems of DEs

Prof. Joyner, 10-24-2007[1]

In this section, we will try to solve the
PROBLEM: Solve

$$\begin{cases} x' = ax + by, & x(0) = x_0, \\ y' = cx + dy, & y(0) = y_0. \end{cases}$$

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In matrix notation, the system of DEs becomes

$$\vec{x}' = A\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{1}$$

where $\vec{x} = \vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

Motivation

First, we shall try to motivate the study of eigenvalues and eigenvectors. This section hopefully will convince you that

- diagonal matrices are wonderful,
- conjugation is very natural,
- if our goal in life is to conjugate a given square matrix matrix into a diagonal one, then eigenvalues and eigenvectors are also natural,
- Solving systems using matrix exponentials.

Diagonal matrices are wonderful: We'll focus for simplicity on the 2×2 case, but everything applies to the general case.

- Addition is easy:

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{pmatrix}.$$

- Multiplication is easy:

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 \cdot b_1 & 0 \\ 0 & a_2 \cdot b_2 \end{pmatrix}.$$

- Powers are easy:

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}^n = \begin{pmatrix} a_1^n & 0 \\ 0 & a_2^n \end{pmatrix}.$$

- You can even exponentiate them:

$$\begin{aligned} exp(t\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \\ &\quad + \frac{1}{2!}t^2\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}^2 + \frac{1}{3!}t^3\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}^3 + \ldots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} ta_1 & 0 \\ 0 & ta_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{1}{2!}t^2a_1^2 & 0 \\ 0 & \frac{1}{2!}t^2a_2^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3!}t^3a_1^3 & 0 \\ 0 & \frac{1}{3!}t^3a_2^3 \end{pmatrix} + \ldots \\ &= \begin{pmatrix} \exp(ta_1) & 0 \\ 0 & \exp(ta_2) \end{pmatrix}. \end{aligned}$$

So, diagonal matrices are wonderful.

Conjugation is natural. You and your friend are piloting a rocket in space. You handle the controls, your friend handles the map. To communicate, you have to "change coordinates". Your coordinates are those of the rocketship (straight ahead is one direction, to the right is another). Your friends coordinates are those of the map (north and east are map directions). Changing coordinates corresponds algebraically to conjugating by a suitable matrix. Using an example, we'll see how this arises in a specific case.

Your basis vectors are

$$v_1 = (1, 0), \quad v_2 = (0, 1),$$

which we call the "v-space coordinates", and the map's basis vectors are

$$w_1 = (1, 1), \quad w_2 = (1, -1),$$

which we call the "w-space coordinates".

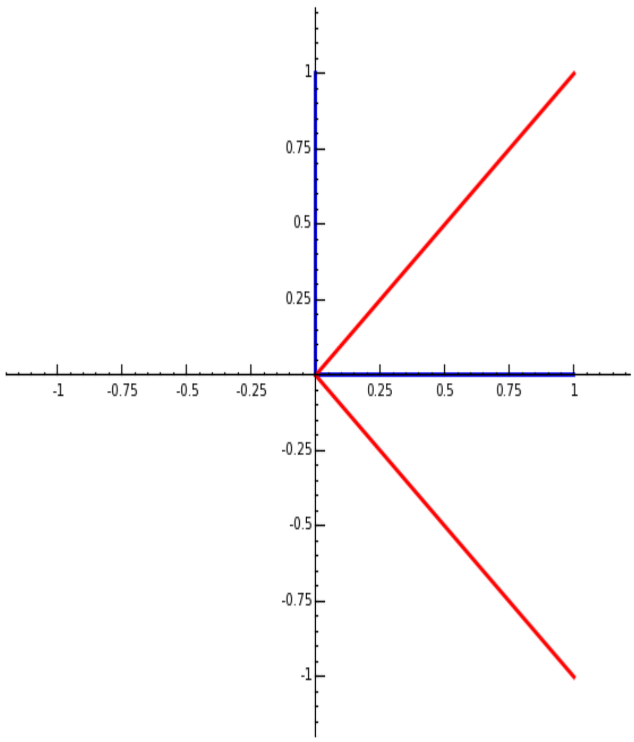

Figure 1: basis vectors v_1, v_2 and w_1, w_2.

For example, the point $(7, 3)$ is, in v-space coordinates of course $(7, 3)$ but in the w-space coordinates, $(5, 2)$ since $5w_1 + 2w_2 = 7v_1 + 3v_2$. Indeed, the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ sends $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$.

Suppose we flip about the 45^o line (the "diagonal") in each coordinate system. In the v-space:

$$av_1 + bv_2 \longmapsto bv_1 + av_2,$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

In other words, in v-space, the "flip map" is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In the w-space:

$$wv_1 + wv_2 \longmapsto aw_1 - bw_2,$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

In other words, in w-space, the "flip map" is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Conjugating by the matrix A converts the "flip map" in w-space to the the "flip map" in v-space:

$$A \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Eigenvalues are natural too: The definition is the following: If A is any square matrix and

$$A\vec{v} = \lambda\vec{v},$$

for some scalar λ (possibly complex) and some *non-zero* vector $\vec{v}$ (also, possibly complex) then λ is called an *eigenvalue* with *eigenvector* $\vec{v}$. The claim is that these are naturally arising objects.

Given a matrix A, is there a basis of the underlying space in which the matrix is diagonal? Given how "wonderful" diagonal matrices are, it seems clear we should find this basis and these diagonal entries.

Fact: When the diagonal entries are distinct, the basis elements are the eigenvectors and the diagonal elements are the eigenvalues.

Since this section is only intended to be motivation, we shall not prove this here (see any text on linear algebra, for example [B] or [H]).

When there is an invertible matrix P and a diagonal matrix D for which $A = P^{-1}DP$ (e.g, when P is the matrix of eigenvectors and D is the diagonal matrix of eigenvalues) then we can compute the matrix exponential. Indeed,

$$\begin{aligned} e^{tA} &= 1 + tA + \frac{1}{2!}(tA)^2 + \frac{1}{3!}(tA)^3 + ... \\ &= P^{-1}P + P^{-1} \cdot tD \cdot P + P^{-1} \cdot \frac{1}{2!}(tD)^2 \cdot P + P^{-1} \cdot \frac{1}{3!}(tD)^3 \cdot P + ... \\ &= P^{-1}(I + +tD + \frac{1}{2!}(tD)^2 + \frac{1}{3!}(tD)^3 + ...)P \\ &= P^{-1}e^{tD}P. \end{aligned}$$

SAGE

```
sage: MS = MatrixSpace(CC,2,2)
sage: A = MS([[0,1],[1,0]])
sage: A.eigenspaces()

[
(1.00000000000000, [
(1.00000000000000, 1.00000000000000)
]),
(-1.00000000000000, [
(1.00000000000000, -1.00000000000000)
])
]
```

This SAGE command tells us that the eigenvalues of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $\lambda = 1, -1$ and the eigenvectors are $\vec{v} = (1, 1), (1, -1)$.

Solving systems of DEs using matrix exponentials.

We know how to take the matrix exponential of a diagonal matrix. Let's assume A is diagonalizable and let

$$\vec{x} = e^{tA}\vec{c}.$$

The derivative with respect to t of this vector-valued function is given by

$$\vec{x}' = Ae^{tA}\vec{c} = A\vec{x}.$$

(This is true even for non-diagonal matrices, and is the analog of the usual derivative formula $\frac{d}{dt}e^{at} = ae^{at}$, where a is a constant.) In other words, $e^{tA}\vec{c}$ solves the system (1). In the next section, we shall see another, more explicit, way of doing this.

Solution strategy

PROBLEM: Solve

$$\begin{cases} x' = ax + by, & x(0) = x_0, \\ y' = cx + dy, & y(0) = y_0. \end{cases}$$

soln: Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In matrix notation, the system of DEs becomes

$$\vec{x}' = A\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

where $\vec{x} = \vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. In a similar manner to how we solved homogeneous constant coefficient 2nd order ODEs $ax'' + bx' + cx = 0$ by using "Euler's guess" $x = Ce^{rt}$, we try to guess an exponential: $\vec{X}(t) = \vec{c}e^{\lambda t}$ (λ is used instead of r to stick with notational convention; $\vec{c}$ in place of C since we need a constant *vector*). Plugging this guess into the matrix DE $\vec{X}' = A\vec{X}$ gives $\lambda\vec{c}e^{\lambda t} = A\vec{c}e^{\lambda t}$, or (cancelling $e^{\lambda t}$)

$$A\vec{c} = \lambda\vec{c}.$$

This means that λ is an eigenvalue of A with eigenvector $\vec{c}$.

- Find the eigenvalues. These are the roots of the characteristic polynomial

  $$p(\lambda) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc).$$

  Call them λ_1, λ_2 (in any order you like).

  You can use the quadratic formula, for example to get them:

  $$\lambda_1 = \frac{a+d}{2} + \frac{\sqrt{(a+d)^2 - 4(ad-bc)}}{2}, \quad \lambda_2 = \frac{a+d}{2} - \frac{\sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

- Find the eigenvectors. If $b \neq 0$ then you can use the formulas

$$\vec{v}_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} b \\ \lambda_2 - a \end{pmatrix}.$$

In general, you can get them by solving the **eigenvector equation** $A\vec{v} = \lambda\vec{v}$.

SAGE

```
sage: MS = MatrixSpace(CC,2,2)
sage: A = MS([[1,2],[3,4]])
sage: A.eigenspaces()

[
(-0.372281323269014, [
(1.00000000000000, -0.457427107756338)
]),
(5.37228132326901, [
(1.00000000000000, 1.45742710775634)
])
]
```

- Plug these into the following formulas:

(a) $\lambda_1 \neq \lambda_2$, real:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\vec{v}_1 \exp(\lambda_1 t) + c_2\vec{v}_2 \exp(\lambda_2 t).$$

(b) $\lambda_1 = \lambda_2 = \lambda$, real:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\vec{v}_1 \exp(\lambda t) + c_2(\vec{v}_1 t + \vec{p}) \exp(\lambda t),$$

where $\vec{p}$ is any non-zero vector satisfying $(A - \lambda I)\vec{p} = \vec{v}_1$.

(c) $\lambda_1 = \alpha + i\beta$, complex: write $\vec{v}_1 = \vec{u}_1 + i\vec{u}_2$, where $\vec{u}_1$ and $\vec{u}_2$ are both *real vectors.*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1[\exp(\alpha t)\cos(\beta t)\vec{u}_1 - \exp(\alpha t)\sin(\beta t)\vec{u}_2]$$
$$+c_2[-\exp(\alpha t)\cos(\beta t)\vec{u}_2 - \exp(\alpha t)\sin(\beta t)\vec{u}_1].$$

Examples

Example 1 *Solve*

$$x'(t)) = x(t) - y(t), \quad y'(t) = 4x(t) + y(t), \quad x(0) = -1, \quad y(0) = 1.$$

Let

$$A = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$$

and so the characteristc polynomial is

$$p(x) = \det(A - xI) = x^2 - 2x + 5.$$

The eigenvalues are

$$\lambda_1 = 1 + 2i, \quad \lambda_2 = 1 - 2i,$$

so $\alpha = 1$ *and* $\beta = 2$*. Eigenvectors* $\vec{v}_1, \vec{v}_2$ *are given by*

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 2i \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ -2i \end{pmatrix},$$

though we actually only need to know $\vec{v}_1$*. The real and imaginary parts of* $\vec{v}_1$ *are*

$$\vec{u}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

The solution is then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -c_1 \exp(t)\cos(2t) + c_2 \exp(t)\sin(2t) \\ -2c_1 \exp(t)\sin(2t) - 2c_2 \exp(t)\cos(2t), \end{pmatrix}$$

so $x(t) = -c_1 \exp(t)\cos(2t) + c_2 \exp(t)\sin(2t)$ *and* $y(t) = -2c_1 \exp(t)\sin(2t) - 2c_2 \exp(t)\cos(2t)$*.*

Since $x(0) = -1$*, we solve to get* $c_1 = 1$*. Since* $y(0) = 1$*, we get* $c_2 = -1/2$*. The solution is:* $x(t) = -\exp(t)\cos(2t) - \frac{1}{2}\exp(t)\sin(2t)$ *and* $y(t) = -2\exp(t)\sin(2t) + \exp(t)\cos(2t)$*.*

Example 2 *Solve*

$$x'(t) = -2x(t) + 3y(t), \quad y'(t) = -3x(t) + 4y(t).$$

Let

$$A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$$

and so the characteristc polynomial is

$$p(x) = \det(A - xI) = x^2 - 2x + 1.$$

The eigenvalues are

$$\lambda_1 = \lambda_2 = 1.$$

An eigenvector $\vec{v}_1$ *is given by*

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Since we can multiply any eigenvector by a non-zero scalar and get another eigenvector, we shall use instead

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let $\vec{p} = \begin{pmatrix} r \\ s \end{pmatrix}$ *be any non-zero vector satisfying* $(A - \lambda I)\vec{p} = \vec{v}_1$. *This means*

$$\begin{pmatrix} -2-1 & 3 \\ -3 & 4-1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

There are infinitely many possibly solutions but we simply take $r = 0ands = 1/3$, *so*

$$\vec{p} = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}.$$

The solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(t) + c_2(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}) \exp(t),$$

or $x(t) = c_1 \exp(t) + c_2 t \exp(t)$ *and* $y(t) = c_1 \exp(t) + \frac{1}{3} c_2 \exp(t) + c_2 t \exp(t)$.

Exercises: Use SAGE to find eigenvalues and eigenvectors of both

$$\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}.$$

References

[B] Robert A. Beezer, **A First Course in Linear Algebra**, released under the GNU Free Documentation License, available at `http://linear.ups.edu/`

[H] Jim Hefferon, **Linear Algebra**, released under the GNU Free Documentation License, available at `http://joshua.smcvt.edu/linearalgebra/`

[S] W. Stein, SAGE, `http://www.sagemath.org/`