

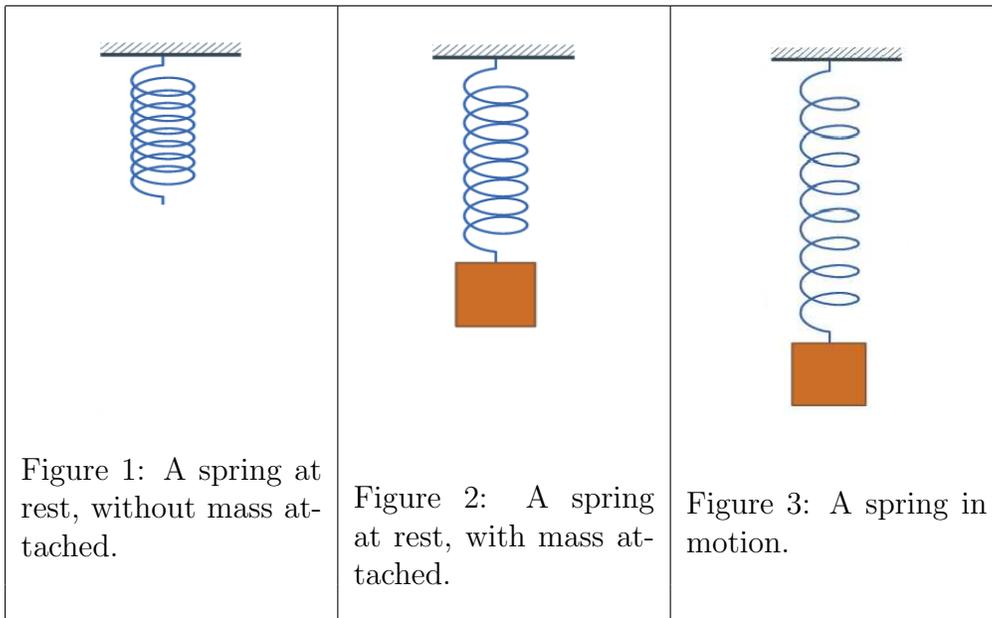
Applications of DEs: Spring problems

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Ut tensio, sic vis².

- *Robert Hooke, 1678*

One of the ways DEs arise is by means of modeling physical phenomenon, such as spring equations. For these problems, consider a spring suspended from a ceiling. We shall consider three cases: (1) no mass is attached at the end of the spring, (2) a mass is attached and the system is in the rest position, (3) a mass is attached and the mass has been displaced from the rest position.



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²“As the extension, so the force.”

One can also align the springs left-to-right instead of top-to-bottom, without changing the discussion below.

Notation: Consider the first two situations above: (a) a spring at rest, without mass attached and (b) a spring at rest, with mass attached. The distance the mass pulls the spring down is sometimes called the “stretch”, and denoted s . (A formula for s will be given later.)

Now place the mass in motion by imparting some initial velocity (tapping it upwards with a hammer, say, and start your timer). Consider the second two situations above: (a) a spring at rest, with mass attached and (b) a spring in motion. The difference between these two positions at time t is called the *displacement* and is denoted $x(t)$. Signs here will be chosen so that down is positive.

Assume exactly three forces act:

1. the restoring force of the spring, F_{spring} ,
2. an external force (driving the ceiling up and down, but may be 0), F_{ext} ,
3. a damping force (imagining the spring immersed in oil or that it is in fact a shock absorber on a car), F_{damp} .

In other words, the total force is given by

$$F_{total} = F_{spring} + F_{ext} + F_{damp}.$$

Physics tells us that the following are approximately true:

1. (Hooke’s law [H]): $F_{spring} = -kx$, for some “spring constant” $k > 0$,
2. $F_{ext} = F(t)$, for some (possibly zero) function F ,
3. $F_{damp} = -bv$, for some “damping constant” $b \geq 0$ (where v denotes velocity),
4. (Newton’s 2nd law [N]): $F_{total} = ma$ (where a denotes acceleration).

Putting this all together, we obtain $mx'' = ma = -kx + F(t) - bv = -kx + F(t) - bx'$, or

$$mx'' + bx' + kx = F(t).$$

This is the **spring equation**. When $b = F(t) = 0$ this is also called the equation for simple harmonic motion.

Consider again first two figures above: (a) a spring at rest, without mass attached and (b) a spring at rest, with mass attached. The mass in the second figure is at rest, so the gravitational force on the mass, mg , is balanced by the restoring force of the spring: $mg = ks$, where s is the stretch. In particular, the spring constant can be computed from the stretch:

$$k = \frac{mg}{s}.$$

Example:

A spring at rest is suspended from the ceiling without mass. A 2 kg weight is then attached to this spring, stretching it 9.8 cm. From a position 2/3 m above equilibrium the weight is given a downward velocity of 5 m/s.

- (a) *Find the equation of motion.*
- (b) *What is the amplitude and period of motion?*
- (c) *At what time does the mass first cross equilibrium?*
- (d) *At what time is the mass first exactly 1/2 m below equilibrium?*

We shall solve this problem using SAGE below. Note $m = 2$, $b = F(t) = 0$ (since no damping or external force is even mentioned), and $k = mg/s = 2 \cdot 9.8 / (0.098) = 200$. Therefore, the DE is $2x'' + 200x = 0$. This has general solution $x(t) = c_1 \cos(10t) + c_2 \sin(10t)$. The constants c_1 and c_2 can be computed from the initial conditions $x(0) = -2/3$ (down is positive, up is negative), $x'(0) = 5$.

Using SAGE, the displacement can be computed as follows:

```

SAGE
sage: t,m,b,k,k,F = var('t,m,b,k,k,F')
sage: x = function('x', t)
sage: de = lambda y: m*diff(y,t,t) + b*diff(y,t) + k*y - F
sage: de(x(t))
-F + m*diff(x(t), t, 2) + b*diff(x(t), t, 1) + k*x(t)
sage: m = 2; b = 0; k = 2*9.8/(0.098); F = 0
sage: de(x(t))

```

```

2*diff(x(t), t, 2) + 200.000000000000*x(t)
sage: desolve(de(x(t)), x)
k1*sin(10*t)+k2*cos(10*t)
sage: desolve_laplace(de(x(t)),["t","x"],[0,-2/3,5])
'sin(10*t)/2-2*cos(10*t)/3'

```

Now we write this in the more compact and useful form $A \sin(\omega t + \phi)$ using the formulas

$$c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \sin(\omega t + \phi),$$

where $A = \sqrt{c_1^2 + c_2^2}$ denotes the *amplitude* and $\phi = 2 \arctan\left(\frac{-2/3}{1/2+A}\right)$.

SAGE

```

sage: A = sqrt((-2/3)^2+(1/2)^2)
sage: A
5/6
sage: phi = 2*atan((-2/3)/(1/2 + A))
sage: phi
-2*atan(1/2)
sage: RR(phi)
-0.927295218001612
sage: sol = lambda t: sin(10*t)/2-2*cos(10*t)/3
sage: sol2 = lambda t: A*sin(10*t + phi)
sage: P = plot(sol(t),0,2)
sage: show(P)

```

This is displayed below³.

³You can also, if you want, type `show(plot(sol2(t),0,2))` to check that these two functions are indeed the same.

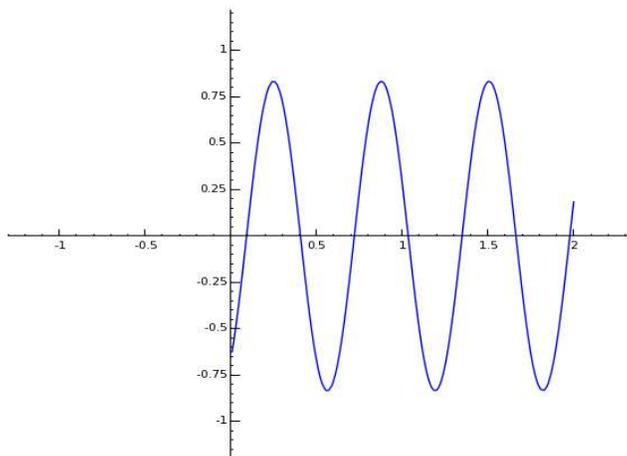


Figure 4: Plot of $2x'' + 200x = 0$, $x(0) = -2/3$, $x'(0) = 5$, for $0 < t < 2$.

Of course, the period is $2\pi/10 = \pi/5 \approx 0.628$.

To answer (c) and (d), we solve $x(t) = 0$ and $x(t) = 1/2$:

SAGE

```
sage: solve(A*sin(10*t + phi) == 0, t)
[t == atan(1/2)/5]
sage: RR(atan(1/2)/5)
0.0927295218001612
sage: solve(A*sin(10*t + phi) == 1/2, t)
[t == (asin(3/5) + 2*atan(1/2))/10]
sage: RR((asin(3/5) + 2*atan(1/2))/10)
0.157079632679490
```

In other words, $x(0.0927\dots) \approx 0$, $x(0.157\dots) \approx 1/2$.

Exercise: *Use the problem above.*

(a) *At what time does the weight pass through the equilibrium position heading down for the 2nd time?*

(b) *At what time is the weight exactly $5/12$ m below equilibrium and heading up?*

References

[H] General wikipedia introduction to Hooke's Law
http://en.wikipedia.org/wiki/Hookes_law

[H1] Wikipedia entry for Robert Hooke:
http://en.wikipedia.org/wiki/Robert_Hooke

[H2] MacTutor entry for Hooke:
<http://www-groups.dcs.st-and.ac.uk/%7Ehistory/Biographies/Hooke.html>

[N] Wikipedia entry for Newton's laws of motion (including Newton's 2nd law): http://en.wikipedia.org/wiki/Newton%27s_laws_of_motion

[S] Wikipedia entry for Simple harmonic motion:
http://en.wikipedia.org/wiki/Simple_harmonic_motion