First order ODEs - separable and linear cases

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Separable DEs:

We know how to solve any ODE of the form

$$y' = f(t),$$

at least in principle - just integrate both sides[2]. For a more general type of ODE, such as

$$y' = f(t, y),$$

this fails. For instance, if $y' = t + y$ then integrating both sides gives $y(t) = \int \frac{dy}{dt}\,dt = \int y'\,dt = \int t + y\,dt = \int t\,dt + \int y(t)\,dt = \frac{t^2}{2} + \int y(t)\,dt$. So, we have only succeeded in writing $y(t)$ in terms of its integral. Not helpful.

However, there is a class of ODEs where this idea works, with some slight modification. If the ODE has the form

$$y' = \frac{g(t)}{h(y)}, \tag{1}$$

then it is called **separable**[3].

To solve a separable ODE:

(1) write the ODE (1) as $\frac{dy}{dt} = \frac{g(t)}{h(y)}$,

[2]Recall y' really denotes $\frac{dy}{dt}$, so by the fundamental theorem of calculus, $y = \int \frac{dy}{dt}\,dt = \int y'\,dt = \int f(t)\,dt = F(t) + c$, where F is the "anti-derivative" of f and c is a constant of integration.

[3]It particular, any separable DE *must* be first order.

(2) "separate" the t's and the y's:

$$h(y)\, dy = g(t)\, dt,$$

(3) integrate both sides:

$$\boxed{\int h(y)\, dy = \int g(t)\, dt \ + \ C}$$

I've added a "$+C$" to emphasize that a constant of integration must be included in your anwser (but only on one side of the equation).

The answer obtained in this manner is called an "implicit solution" of (1) since it expresses y *implicitly* as a function of t.

Example 1. *Are the following ODEs separable? If so, solve them.*

(a) $(t^2 + y^2)y' = -2ty$,

(b) $y' = -x/y$, $y(0) = -1$,

(c) $T' = k \cdot (T - T_{room})$, *where* $k < 0$ *and* T_{room} *are constants,*

(d) $ax' + bx = c$, *where* $a \neq 0$, $b \neq 0$, *and* c *are constants*

(e) $ax' + bx = c$, *where* $a \neq 0$, b, *are constants and* $c = c(t)$ *is not a constant.*

(f) $y' = (y - 1)(y + 1)$, $y(0) = 2$.

(g) $y' = y^2 + 1$, $y(0) = 1$.

Solutions*:*

(a) *not separable,*

(b) $y\,dy = -x\,dx$, *so* $y^2/2 = -x^2/2 + c$, *so* $x^2 + y^2 = 2c$. *This is the general solution (note it does not give* y *explicitly as a function of* x*, you will have to solve for* y *algebraically to get that). The initial conditions say when* $x = 0$, $y = 1$, *so* $2c = 0^2 + 1^2 = 1$, *which gives* $c = 1/2$. *Therefore,* $x^2 + y^2 = 1$, *which is a circle. That is not a function so cannot be the solution we want. The solution is either* $y = \sqrt{1 - x^2}$ *or* $y = -\sqrt{1 - x^2}$, *but which one? Since* $y(0) = -1$ *(note the minus sign) it must be* $y = -\sqrt{1 - x^2}$.

(c) $\frac{dT}{T - T_{room}} = kdt$, *so* $\ln|T - T_{room}| = kt + c$ *(some constant* c*), so* $T - T_{room} = Ce^{kt}$ *(some constant* C*), so* $T = T(t) = T_{room} + Ce^{kt}$.

(d) $\frac{dx}{dt} = (c - bx)/a = -\frac{b}{a}(x - \frac{c}{b})$, *so* $\frac{dx}{x - \frac{c}{b}} = -\frac{b}{a}\,dt$, *so* $\ln|x - \frac{c}{b}| = -\frac{b}{a}t + C$, *where* C *is a constant of integration. This is the implicit general solution of the DE. The explicit general solution is* $x = \frac{c}{b} + Be^{-\frac{b}{a}t}$, *where* B *is a constant.*

The explicit solution is easy find using SAGE *:*

SAGE

```
sage: a = var('a')
sage: b = var('b')
sage: c = var('c')
sage: t = var('t')
sage: x = function('x', t)
sage: de = lambda y: a*diff(y,t) + b*y - c
sage: desolve_laplace(de(x(t)),["t","x"])
'c/b-(a*c-x(0)*a*b)*%e^-(b*t/a)/(a*b)'
```

(e) *If* $c = c(t)$ *is not constant then* $ax' + bx = c$ *is not separable.*

(f) $\frac{dy}{(y-1)(y+1)} = dt$ *so* $\frac{1}{2}(\ln(y-1) - \ln(y+1)) = t + C$, *where* C *is a constant of integration. This is the "general (implicit) solution" of the DE.*

Note: the constant functions $y(t) = 1$ *and* $y(t) = -1$ *are also solutions to this DE. These solutions cannot be obtained (in an obvious way) from the general solution.*

The integral is easy to do using SAGE *:*

SAGE

```
sage: y = var('y')
sage: integral(1/((y-1)*(y+1)),y)
log(y - 1)/2 - (log(y + 1)/2)
```

Now, let's try to get SAGE *to solve for* y *in terms of* t *in* $\frac{1}{2}(\ln(y-1) - \ln(y+1)) = t + C$:

SAGE

```
sage: C = var('C')
sage: solve([log(y - 1)/2 - (log(y + 1)/2) == t+C],y)
[log(y + 1) == -2*C + log(y - 1) - 2*t]
```

This is not working. Let's try inputting the problem in a different form:

SAGE

```
sage: C = var('C')
sage: solve([log((y - 1)/(y + 1)) == 2*t+2*C],y)
[y == (-e^(2*C + 2*t) - 1)/(e^(2*C + 2*t) - 1)]
```

This is what we want. Now let's assume the initial condition $y(0) = 2$ and solve for C and plot the function.

SAGE

```
sage: solny=lambda t:(-e^(2*C+2*t)-1)/(e^(2*C+2*t)-1)
sage: solve([solny(0) == 2],C)
[C == log(-1/sqrt(3)), C == -log(3)/2]
sage: C = -log(3)/2
sage: solny(t)
(-e^(2*t)/3 - 1)/(e^(2*t)/3 - 1)
sage: P = plot(solny(t), 0, 1/2)
sage: show(P)
```

This plot is shown below. The solution has a singularity at $t = ln(3)/2 = 0.5493....$

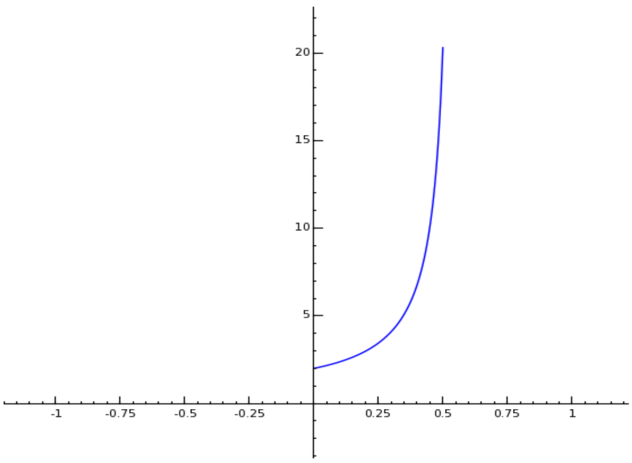

Figure 1: Plot of $y' = (y-1)(y+1)$, $y(0) = 2$, for $0 < t < 1/2$..

(g) $\frac{dy}{y^2+1} = dt$ *so* $\arctan(y) = t + C$, *where C is a constant of integration. The initial condition $y(0) = 1$ says* $\arctan(1) = C$, *so $C = \frac{\pi}{4}$. Therefore $y = \tan(t + \frac{\pi}{4})$ is the solution.*

A special subclass of separable ODEs is the class of **automomous** ODEs, which have the form

$$y' = f(y),$$

where f is a given function (i.e., the slope y only depends on the value of the dependent variable y). The cases (c), (d), (f), and (g) above are examples.

Linear 1st order DEs:

The bottom line is that we want to solve any problem of the form

$$x' + p(t)x = q(t), \tag{2}$$

where $p(t)$ and $q(t)$ are given functions (which, let's assume, aren't too horrible). Every first order linear ODE can be written in this form. Examples of DEs which have this form: Falling Body problems, Newton's Law of Cooling problems, Mixing problems, certain simple Circuit problems, and so on.

There are two approaches

- "the formula",
- the method of integrating factors.

Both lead to the exact same solution.
"The Formula": The general solution to (2) is

$$x = \frac{\int e^{\int p(t)\,dt} q(t)\,dt + C}{e^{\int p(t)\,dt}}, \tag{3}$$

where C is a constant. The factor $e^{\int p(t)\,dt}$ is called the **integrating factor** and is often denoted by μ. This formula was apparently first discovered by Johann Bernoulli [F].

Example 2. *Solve*

$$xy' + y = e^x.$$

We rewrite this as $y' + \frac{1}{x}y = \frac{e^x}{x}$*. Now compute* $\mu = e^{\int \frac{1}{x}\,dx} = e^{\ln(x)} = x$*, so the formula gives*

$$y = \frac{\int x\frac{e^x}{x}\,dx + C}{x} = \frac{\int e^x\,dx + C}{x} = \frac{e^x + C}{x}.$$

Here is one way to do this using SAGE *:*

SAGE

```
sage: t = var('t')
sage: x = function('x', t)
sage: de = lambda y: diff(y,t) + (1/t)*y - exp(t)/t
sage: desolve(de(x(t)),[x,t])
'(%e^t+%c)/t'
```

"Integrating factor method": Let $\mu = e^{\int p(t)\,dt}$. Multiply both sides of (2) by μ:

$$\mu x' + p(t)\mu x = \mu q(t).$$

The product rule implies that

$$(\mu x)' = \mu x' + p(t)\mu x = \mu q(t).$$

(In response to a question you are probably thinking now: No, this is not obvious. This is Bernoulli's very clever idea.) Now just integrate both sides. By the fundamental theorem of calculus,

$$\mu x = \int (\mu x)' \, dt = \int \mu q(t) \, dt.$$

Dividing both side by μ gives (3).

Exercise: (a) Use SAGE 's `desolve` command to solve

$$tx' + 2x = e^t/t.$$

(b) Use SAGE to plot the solution to $y' = y^2 - 1$, $y(0) = -2$.

References

[BD] W. Boyce and R. DiPrima, **Elementary Differential Equations and Boundary Value Problems**, 8th edition, John Wiley and Sons, 2005.

[F] General wikipedia introduction to First order linear differential equations:
`http://en.wikipedia.org/wiki/Linear_differential_equation#First_order_equation`