FourierSeriesAndSage

```
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%html
Joseph Fourier (1768 – 1830)
<img src="http://upload.wikimedia.org/wikipedia/commons
/4/49/Joseph_Fourier_%28circa_1820%29.jpg">
```

Joseph Fourier (1768 – 1830)

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Fourier series were introduced by Joseph Fourier, a French physicist. In fact, Fourier
was Napolean's scientific advisor during France's invasion of
Egypt in the late 1700's. When Napolean returned to France,
he ``elected'' (i.e., appointed) Fourier to be a Prefect - basically
an important administrative post where he oversaw some large
construction projects, such as highway constructions. It was during this
time that Fourier worked on the theory of heat on the side.

However, Fourier's solution, explained in his
1807 memoir {\it On the Propagation of Heat in Solid Bodies},
was awarded a mathematics prize in 1811 by the
Paris Institute. We shall discuss this solution later.

First, we define Fourier series and give some simpler applications.
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First, we define Fourier series and give some simpler applications.

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\begin{center}
{\bf \LARGE{Introduction to Fourier series - use as approximations}}
\end{center}
```

Introduction to Fourier series - use as approximations

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```
The Taylor series approximation of a given function re-expresses that function as a
linear combination of ``simpler functions'', namely powers $x^n$, $n=0,1,2,\dots$.
Using Sage's {\tt taylor} method, you can easily compute the Taylor series
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The Taylor series approximation of a given function re-expresses that function as a linear combination of “simpler functions”, namely powers x^n, $n = 0, 1, 2, \dots$. Using Sage’s **taylor** method, you can easily compute the Taylor series approximation to many common functions, such as the exponential function:

```
x = var("x")
f = e^x
f.taylor(x,0,3)
```

```
1/6*x^3 + 1/2*x^2 + x + 1
```

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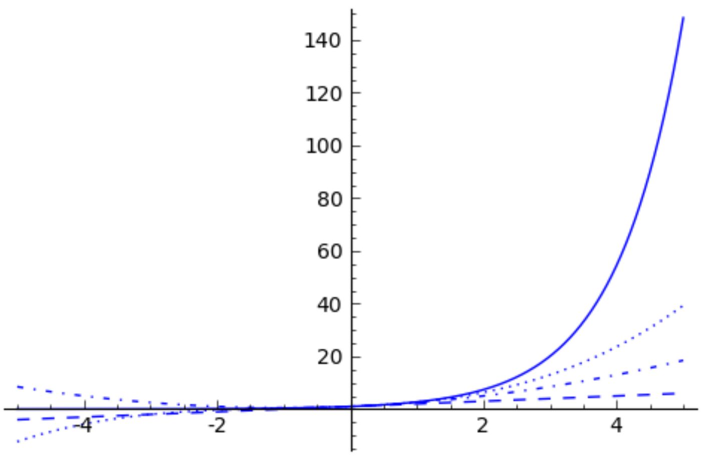
We can also use Sage to plot these approximations, to see visually how they get better
and better:
```

We can also use Sage to plot these approximations, to see visually how they get better and better:

```
P1 = plot(e^x, x, -5, 5)
P2 = plot(1+x, x, -5, 5, linestyle="--")
P3 = plot(1+x+x^2/2, x, -5, 5, linestyle="-.")
P4 = plot(1+x+x^2/2+x^3/6, x, -5, 5, linestyle=":")
(P1+P2+P3+P4).show()
```




```
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Just as the Taylor series re-expresses a given function as a linear combination of
powers
\[
x^n, \ \ \ \ \ \ \ \ \ \ n = 0, 1, 2, \dots ,
\]
the Fourier series re-expresses a given function as a linear combination of cosines and
sines:

\[
    a_n\cos(\frac{n\pi t}{L})+b_n\sin(\frac{n\pi t}{L}),\ \ \ \ \ \ \ \ \ \ n = 0, 1,
2, \dots .
\]
The Fourier series is applied to periodic functions (of period $2L$, where $L>0$ is any
number you choose).
For example, if you choose $L=\pi$ and the function to be

\[
f(x)=
```

```
\left\{
\begin{array}{ll}
-1, & -\pi<x<0,\\
2, & 0<x<\pi/2,\\
1, & \pi/2 < x< \pi,
\end{array}
\right.
\]
then we can use Sage to compute the Fourier series as follows.
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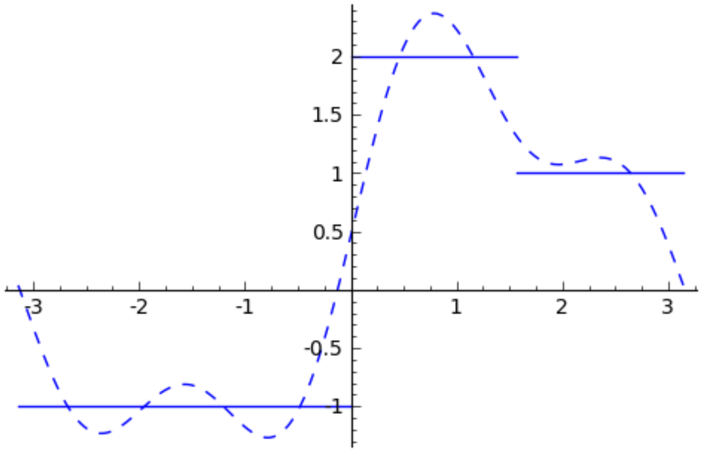
The Fourier series is applied to periodic functions (of period $2L$, where $L > 0$ is any number you choose). For example, if you choose $L = \pi$ and the function to be

$$f(x) = \left\{ \begin{array}{ll} -1, & -\pi<x<0, \\ 2, & 0<x<\pi/2, \\ 1, & \pi/2 < x < \pi, \end{array} \right.$$

then we can use Sage to compute the Fourier series as follows.

```
f1(x) = -1
f2(x) = 2
f3(x) = 1
f = Piecewise([[(-pi,0),f1],[(0,pi/2),f2],[(pi/2,pi),f3]])
n = var("n")
L = pi
an = (1/L)*integral(f1(x)*cos(n*pi*x/L),x,-pi,0)+(1/L)*integral(f2(x)*cos(n*pi*x
/L),x,0,pi/2)+(1/L)*integral(f3(x)*cos(n*pi*x/L),x,pi/2,pi)
print "a_n = ",an
bn = (1/L)*integral(f1(x)*sin(n*pi*x/L),x,-pi,0)+(1/L)*integral(f2(x)*sin(n*pi*x
/L),x,0,pi/2)+(1/L)*integral(f3(x)*sin(n*pi*x/L),x,pi/2,pi)
print "b_n = ",bn
FS5 = f.fourier_series_partial_sum(5,pi)
print "partial FS = ",FS5
P1 = f.plot()
P2 = plot(FS5, x, -pi, pi, linestyle="--")
(P1+P2).show()
```

```
a_n =  -(sin(1/2*pi*n)/n - sin(pi*n)/n)/pi + 2*sin(1/2*pi*n)/(pi*n)
- sin(pi*n)/(pi*n)
b_n =  -(cos(pi*n)/n - 1/n)/pi - 2*(cos(1/2*pi*n)/n - 1/n)/pi +
(cos(1/2*pi*n)/n - cos(pi*n)/n)/pi
partial FS =  sin(2*x)/pi + 5/3*sin(3*x)/pi + 5*sin(x)/pi -
1/3*cos(3*x)/pi + cos(x)/pi + 1/4
```




```
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Though our Fourier series approximation to our piecewise constant function $f(x)$ has
$5$ terms,

\[
\frac{1}{4}+\frac{1}{\pi}\cos(x) + \frac{5}{\pi}\sin(x)
\frac{1}{\pi}\sin(2 x)  - \frac{1}{3\pi}\cos(3x)+ \frac{5}{3\pi}\sin(3x),
\]
we can see from the graph that it is a relatively bad approximation. Let us see what
happens when we pick more terms.
```

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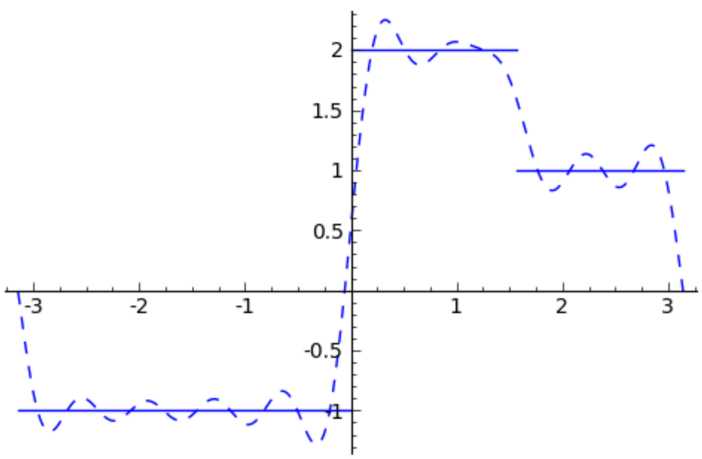
we can see from the graph that it is a relatively bad approximation. Let us see what happens when we pick more terms.

```
FS10 = f.fourier_series_partial_sum(10,pi)
P3 = plot(FS10, x, -pi, pi, linestyle="--")
(P1+P3).show()
```




```
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This approximation is better. But for $13$ terms,

\[
```

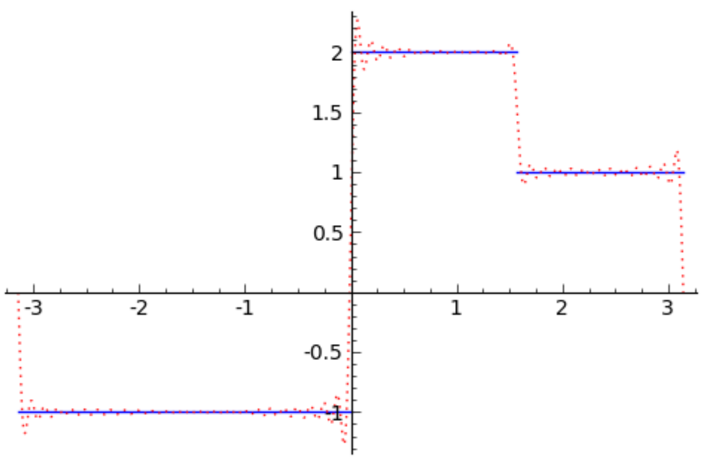
```
 \frac{1}{4}
+
\frac{\cos\left(x\right)}{\pi}
+
\frac{5 \, \sin\left(x\right)}{\pi}
+
\frac{\sin\left(2 \, x\right)}{\pi}
- \frac{\cos\left(3 \, x\right)}{3\, \pi}
+ \frac{5 \, \sin\left(3 \,x\right)}{3 \, \pi}
+ \frac{\cos\left(5 \, x\right)}{5 \, \pi}
+ \frac{\sin\left(5 \, x\right)}{\pi} +
\frac{\sin\left(6 \, x\right)}{3 \, \pi}
- \frac{\cos\left(7\, x\right)}{7 \, \pi}
+ \frac{5 \, \sin\left(7 \,x\right)}{7 \, \pi}
+ \frac{5 \, \sin\left(9 \, x\right)}{9 \, \pi}
 + \frac{\cos\left(9 \, x\right)}{9 \, \pi} ,
\]
it still doesn't seem worth the effort! Let's try more terms.
```

This approximation is better. But for 13 terms,

$$\frac{1}{4}+\frac{\cos\left(x\right)}{\pi}+\frac{5\,\sin\left(x\right)}{\pi}+\frac{\sin\left(2\,x\right)}{\pi}-\frac{\cos\left(3\,x\right)}{3\,\pi}+\frac{5\,\sin\left(3\,x\right)}{3\,\pi}+\frac{\cos\left(5\,x\right)}{5\,\pi}+\frac{\sin\left(5\,x\right)}{\pi}+\frac{\sin\left(6\,x\right)}{3\,\pi}-\frac{\cos\left(7\,x\right)}{7\,\pi}+\frac{5\,\sin\left(7\,x\right)}{7\,\pi}$$

it still doesn't seem worth the effort! Let's try more terms.

```
FS50 = f.fourier_series_partial_sum(50,pi)
P4 = plot(FS50, x, -pi, pi, linestyle=":",rgbcolor=(1,0,0))
(P1+P4).show()
```




```
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Now, the approximation is so good, the Fourier series approximation plot uses red dots
for added contrast to help you see how much better it is.

\par
Clearly, these Fourier series approximations are getting better and better, the more
and more terms we take. However, they do require a lot of terms to get a good
approximation and that is a problem. There are methods to help get around that (using
``Cesaro filters'', for example) but the techniques are a little too advanced for this
introduction. We will give an example using Sage but refer to a book on signal
processing for details.
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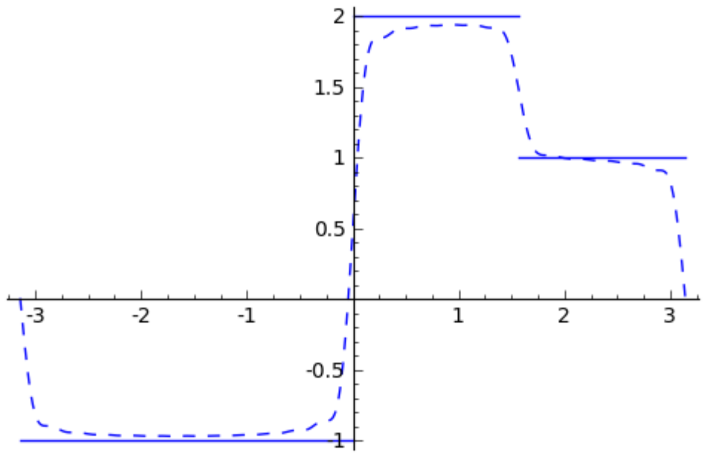
```
FS25C = f.fourier_series_partial_sum_cesaro(25,pi)
P3C = plot(FS25C, x, -pi, pi, linestyle="--")
(P1+P3C).show()
```

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\begin{center}
{\bf \LARGE{Application of Fourier series - to the physics of sound}}
\end{center}
```

Application of Fourier series - to the physics of sound

```
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Play a note on some musical instrument, for example a flute. The sound produced is
heard by the human ear
as a result of the sound waves traveling through the air and vibrating our eardrums.
Plot the deviations from average air pressure as a function of time. Such graphs are
called
waveforms. ({\small{A small technicality: what we call a waveform is usually referred
to as
a normalized waveform - one where the phase shifts have been normalized in a
certain way - see [1] for details}}.)

\par
Let $f(t)$ denote that waveform. It is (ideally) periodic with some period $P>0$. (For
example, in the case
of the saxophone playing the middle C note, we could take $P=1/262$.) Write down $f(t)$
as
a Fourier series with period $2L$. In doing so, we are expressing the sound as a sum of
``simple pure sounds''

\[
a_n\cos(\frac{n\pi t}{L})+b_n\sin(\frac{n\pi t}{L}),\ \ \ \ \ \ \ \ \ \ n = 0, 1, 2,
\dots .
```

```
\]
This $n$th term of the Fourier series is called the $n$th {\it harmonic} of $f$.
The {\it amplitude} of the $n$th harmonic is the quantity

\[
\sqrt{a_n^2+b_n^2}.
\]

References:

[1] Saxophone waveforms
\verb+http://www.phy.mtu.edu/~suits/sax_sounds/index.html+
 and
\verb+http://www.phy.mtu.edu/~suits/phaseshifts.html+
```

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Let $f(t)$ denote that waveform. It is (ideally) periodic with some period $P > 0$. (For example, in the case of the saxophone playing the middle C note, we could take $P = 1/262$.) Write down $f(t)$ as a Fourier series with period $2L$. In doing so, we are expressing the sound as a sum of "simple pure sounds"

$$a_n \cos(\frac{n\pi t}{L}) + b_n \sin(\frac{n\pi t}{L}), \qquad n = 0, 1, 2, \ldots.$$

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$$\sqrt{a_n^2 + b_n^2}.$$

References:

[1] Saxophone waveforms http://www.phy.mtu.edu/~suits/sax_sounds/index.html and http://www

```
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We use Sage to plot a finite Fourier series which approximates the flute A4 note and
whose amplitudes match those in the graph given in [2].

References:

[2] Flute (normalized) waveform:
\verb+http://hyperphysics.phy-astr.gsu.edu/hbase/music/flutew.html#c1+

See below.
```

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See below.

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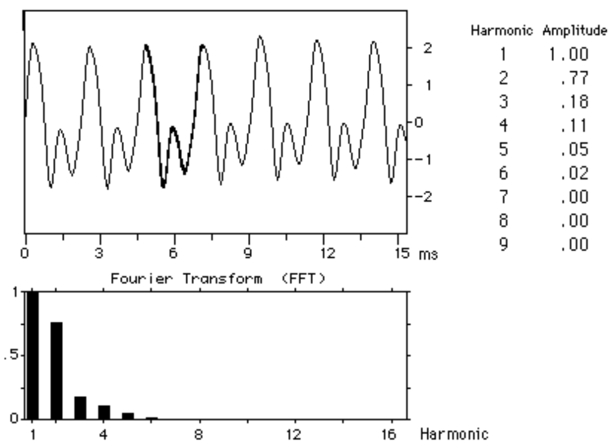
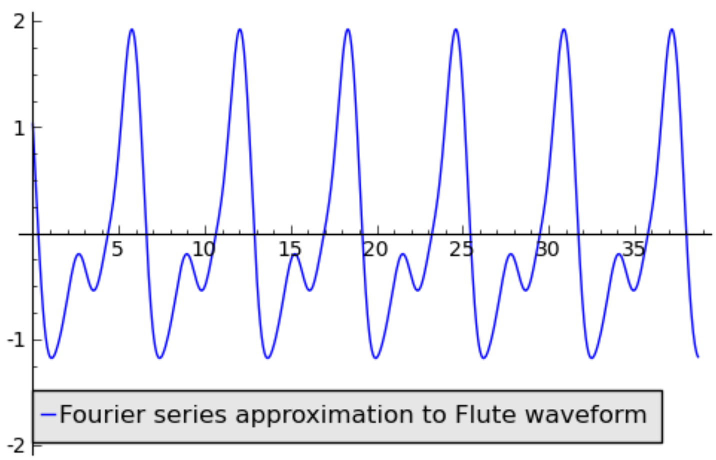
<b>
```

```
Flute waveform, A4 :
</b>
<p>
<img src="http://hyperphysics.phy-astr.gsu.edu/hbase/music/imgmus/fluw3.gif" alt="Flute
waveform, A4" />
```

Flute waveform, A4 :




```
g(x) =
0.8*sin(x)+0.6*cos(x)+0.7*sin(2*x)+0.32*cos(2*x)+0.14*sin(3*x)+0.11*cos(3*x)++0.11*sin(4*x)
plot(g(-x),(x,0,12*pi+1),ymin=-2,ymax=2,legend_label="Fourier series approximation to
Flute waveform")
```

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EXERCISE: Do the same thing for the saxophone C4 note, shown below.

<p>

    <b>
Saxophone waveform, middle C :
</b>
<p>
<img src="http://hyperphysics.phy-astr.gsu.edu/hbase/music/imgmus/saxw1.gif"
alt="Saxophone waveform, middle C" />
<p>

Reference:
```

```
    Saxophone (normalized) waveform:
http://hyperphysics.phy-astr.gsu.edu/hbase/music/saxw.html
```

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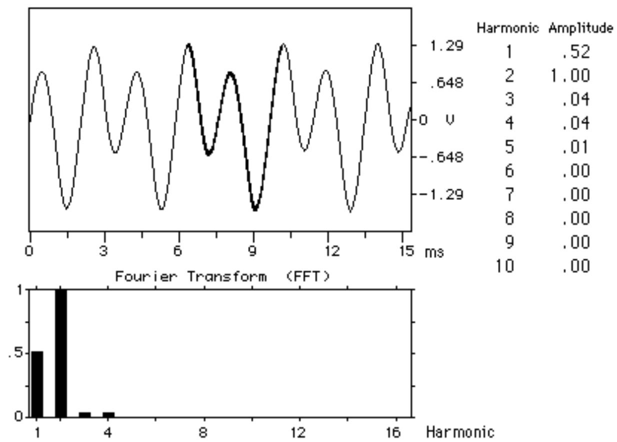
Saxophone waveform, middle C :



Reference: Saxophone (normalized) waveform: http://hyperphysics.phy-astr.gsu.edu/hbase/music/saxw.html

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\begin{center}
{\bf \LARGE{Application of Fourier series - to the evaluation of $\zeta(2)$}}
\end{center}
```

Application of Fourier series - to the evaluation of $\zeta(2)$

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The Riemann hypothesis is one of the \$ 1,000,000 Millennium Prize Problems, sponsored
by the Clay Mathematics Institute [1].

A math problem worth a million dollars? Yes, and here is one way to define it [2].

The Riemann zeta function is defined for complex $s$ with real part
greater than $1$
by the absolutely convergent infinite series

\[
    \zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} +
\frac{1}{3^s} + \cdots.
\]
Leonhard Euler showed that this series equals the Euler product

\[
    \zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}
= \frac{1}{1-2^{-s}}\cdot\frac{1}{1-3^{-s}}\cdot\frac{1}{1-5^{-s}}\cdot\frac{1}
{1-7^{-s}} \cdots \frac{1}{1-p^{-s}} \cdots
\]
where the infinite product extends over all prime numbers $p$.
Again, this converges for complex $s$ with real part greater than
    $1$.
The convergence of the Euler product shows that $\zeta(s)$ has no zeros in this region,
as none of the factors have zeros.

The Riemann hypothesis discusses zeros for values of $s$ in range
where the real part of $s$ lies between $0$ and $1$, so it needs to be
```

```
analytically continued to a larger
domain in the complex plane. This can be done as follows. If $s$ has
real part greater than one, then the zeta function satisfies

\[
    \left(1-\frac{2}{2^s}\right)\zeta(s) = \sum_{n=1}^\infty
    \frac{(-1)^{n+1}}{n^s}
= \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \, .
\]
However, the series on the right converges not just when $s$ has real
    part greater than one, but more generally whenever $s$ has positive real part. Thus,
this alternative series extends the zeta function from $ Re(s) > 1$ to
the larger domain $Re(s) > 0$.

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{\large{
The {\bf Riemann hypothesis} states that:
{\it If $s$ is any zero of the Riemann zeta function, $\zeta(s)=0$, with
$0<Re(s)<1$ then $Re(s)=1/2$. }
}}
\vskip .3in

References:

    [1] Millennium Prize Problems

\verb+http://en.wikipedia.org/wiki/Millennium_Prize_Problems+

\verb+http://en.wikipedia.org/wiki/Clay_Mathematics_Institute#Millennium_Prize_Problems+

[2] Riemann hypothesis
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Leonhard Euler showed that this series equals the Euler product

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \frac{1}{1-7^{-s}} \cdots \frac{1}{1-p^{-s}} \cdots$$

where the infinite product extends over all prime numbers p. Again, this converges for complex s with real part greater than 1. The convergence of the Euler product shows that $\zeta(s)$ has no zeros in this region, as none of the factors have zeros.

The Riemann hypothesis discusses zeros for values of s in range where the real part of s lies between 0 and 1, so it needs to be analytically continued to a larger domain in the complex plane. This can be done as follows. If s has real part greater than one, then the zeta function satisfies

$$\left(1 - \frac{2}{2^s}\right) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots .$$

However, the series on the right converges not just when s has real part greater than one, but more generally whenever s has positive real part. Thus, this alternative series extends the zeta function from $Re(s) > 1$ to the larger domain $Re(s) > 0$.

The **Riemann hypothesis** states that: *If s is any zero of the Riemann zeta function, $\zeta(s) = 0$, with $0 < Re(s) < 1$ then $Re(s) = 1/2$.*

References:

[1] Millennium Prize Problems
http://en.wikipedia.org/wiki/Millennium_Prize_Problems
http://en.wikipedia.org/wiki/Clay_Mathematics_Institute#Millennium_Prize_Problems
[2] Riemann hypothesis http://en.wikipedia.org/wiki/Riemann_hypothesis

```
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We will use Fourier series to evaluate the Riemann zeta function at $s=2$.

One derivation for $\zeta(2)$ considers the Fourier series of
$f(x)=x^2$, using $L=1$ (so the period is $2$).

\[
f(x)=a_0/2+\sum_{n=1}^\infty [a_n\cos(n\pi x)+b_n\sin(n\pi x)],
\]
which has coefficients given by

\[
a_0     =       2/3,\ \ \ \
a_n     =       \frac{4\cos(\pi n)}{\pi^2}n^{-2},\ \ \ \
b_n     =       0,
\]
for $n>0$, so the Fourier series is

\[
```

```
x^2=1/3+4\sum_{n=1}^\infty \frac{\cos(n\pi)}{\pi^2n^2}\cos(n\pi x).
\]
Letting $x=1$ and noting that $\cos(n\pi)^2=1$, gives

\[
2/3 = 4\sum_{n=1}^\infty \frac{1}{\pi^2n^2}.
\]
Divide by $4/\pi^2$ and we have
\[
\zeta(2)=\sum_{n=1}^\infty n^{-2}
=\frac{\pi^2}{6}.
\]

Here is a short Sage computation to compute the $a_n$'s:
```

We will use Fourier series to evaluate the Riemann zeta function at $s = 2$.
One derivation for $\zeta(2)$ considers the Fourier series of $f(x) = x^2$, using $L = 1$ (so the period is 2).

$$f(x) = a_0/2 + \sum_{n=1}^{\infty}[a_n \cos(n\pi x) + b_n \sin(n\pi x)],$$

which has coefficients given by

$$a_0 = 2/3, \qquad a_n = \frac{4\cos(\pi n)}{\pi^2} n^{-2}, \qquad b_n = 0,$$

for $n > 0$, so the Fourier series is

$$x^2 = 1/3 + 4\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\pi^2 n^2} \cos(n\pi x).$$

Letting $x = 1$ and noting that $\cos(n\pi)^2 = 1$, gives

$$2/3 = 4\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2}.$$

Divide by $4/\pi^2$ and we have

$$\zeta(2) = \sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}.$$

Here is a short Sage computation to compute the a_n's:

```
x = var("x")
n = var("n")
f1(x) = x^2
L = 1
a0 = (2/L)*integrate(f1(x)*cos(pi*x*0/L), (x, 0, L))
an = (2/L)*integrate(f1(x)*cos(pi*x*n/L), (x, 0, L))
a0; an.expand()
```

```
2/3
2*sin(pi*n)/(pi*n) + 4*cos(pi*n)/(pi^2*n^2) - 4*sin(pi*n)/(pi^3*n^3)
```

```
%hide
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\begin{center}
{\bf \LARGE{Application of Fourier series - to solving the heat equation $u_t=ku_{xx}$}}
\end{center}
```

Application of Fourier series - to solving the heat equation $u_t = ku_{xx}$

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%hide
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This final sections shows how Sage can be used to illustrate Fourier's solution to the
heat equation.

\par
The heat equation is the PDE

\[
k\frac{\partial^2u(x,t)}{ \partial x^2}
= \frac{\partial u(x,t)}{ \partial t}.
\]
This is a diffusion
equation whose study is very important for many
topics (even in subjects as diverse as financial mathematics).

The heat equation with {\bf zero ends} boundary conditions
models the temperature of an (insulated) wire of length $L$:

\[
\left\{
\begin{array}{c}
k\frac{\partial^2u(x,t)}{ \partial x^2}
= \frac{\partial u(x,t)}{ \partial t},\\
u(x,0)=f(x),\\
u(0,t)=u(L,t)=0.\\
\end{array}
\right.
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Here $u(x,t)$ denotes the temperature at a point
$x$ on the wire at time $t$. The initial temperature
$f(x)$ is a given more-or-less arbitrary function
that is related to our temperature function
$u(x,t)$ by the condition:

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In this model, it is assumed no heat escapes out the middle of the
wire (which, say, is coated with some kind of insulating plastic).
However, the boundary conditions, $u(0,t)=u(L,t)=0$ permits heat to
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The heat equation with **zero ends** boundary conditions models the temperature of an (insulated) wire of length L:

$$\begin{cases} k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}, \\ u(x,0) = f(x), \\ u(0,t) = u(L,t) = 0. \end{cases}$$

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\begin{center}
{\bf Method for zero ends}
\end{center}
The ``recipe'' for solving a zero ends heat problem by Fourier's method
is pretty simple.

\begin{itemize}

\item
Find the sine series ($SS_f(x)$) of $f(x)$:

\[
f(x)\sim \sum_{n=1}^\infty
b_n(f)\sin(\frac{n\pi x}{ L}),
\]
where the coefficients can be computed from

\[
b_n=\frac{2}{L}\int_0^L \sin(\frac{n\pi x}{L})f(x)\ dx.
\]
(The $\sim$ notation means that the sine series $SS_f(x)$ converges to
$f(x)$, if $f(x)$ is continuous at $x$, and to the ``midpoint of the jump''
$\frac{1}{2}(f(x+)+f(x-))$, if $f(x)$ has a jump discontinuity at $x$.)

\item
The solution is


\[
u(x,t)=\sum_{n=1}^\infty
b_n(f)\sin(\frac{n\pi x}{L})
\exp(-k(\frac{n\pi}{L})^2t).
\]
(A quick reminder: $exp(x)=e^x$.)

\end{itemize}
```

```
Here is a quick explanation using separation of variables and the superposition
principle.
(See a textbook such as Boyce+DiPrima for more details.)
First, use the sine series for $f(x)$ to rewrite the problem as

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Next, using the superposition principle, we reduce this problem down to solving
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for $n=1,2,\dots$. Using separation of variables, we try to
find a solution to this problem in the form $u(x,t)=X(x)T(t)$.
By plugging this into the PDE, separating variables and then using the
other conditions, we find $X(x)T(t)=b_n(f)\sin(\frac{n\pi x}{L})
\exp(-k(\frac{n\pi}{L})^2t)$, as desired.
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Method for zero ends

The "recipe" for solving a zero ends heat problem by Fourier's method is pretty simple.

- Find the sine series ($SS_f(x)$) of $f(x)$:

$$f(x) \sim \sum_{n=1}^{\infty} b_n(f) \sin(\frac{n\pi x}{L}),$$

  where the coefficients can be computed from

$$b_n = \frac{2}{L}\int_0^L \sin(\frac{n\pi x}{L}) f(x)\ dx.$$

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- The solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(f) \sin(\frac{n\pi x}{L}) \exp(-k(\frac{n\pi}{L})^2 t).$$

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Here is a quick explanation using separation of variables and the superposition principle. (See a textbook such as Boyce+DiPrima for more details.) First, use the sine series for $f(x)$ to rewrite the problem as

$$\left\{ \begin{array}{c} k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}, \\ u(x,0) = \sum_{n=1}^{\infty} b_n(f)\sin(\frac{n\pi x}{L}), \\ u(0,t) = u(L,t) = 0. \end{array} \right.$$

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for $n = 1, 2, \ldots$. Using separation of variables, we try to find a solution to this problem in the form $u(x,t) = X(x)T(t)$. By plugging this into the PDE, separating variables and then using the other conditions, we find $X(x)T(t) = b_n(f)\sin(\frac{n\pi x}{L})\exp(-k(\frac{n\pi}{L})^2 t)$, as desired.

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{\bf Example}:
Let

\[
f(x)=
\left\{
\begin{array}{ll}
-1,& 0\leq x\leq \pi/2, \\
2, & \pi/2<x<\pi.
\end{array}
\right.
\]
Then $L=\pi$ and

\[
```

```
b_n(f)=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(n x)dx=
-2\,{\frac {2\,\cos(n\pi )-3\,\cos(\frac{1}{2}\,n\pi )+1}{n\pi}}.
\]
Thus

\[
f(x)\sim b_1(f)\sin(x)+b_2(f)\sin(2x)+...
=\frac{2}{\pi}\sin(x)-\frac{6}{\pi}\sin(2x)+\frac{2}{3\pi}\sin(3x)+... .
\]

This can also be done in Sage. First, we compute and plot several terms of the sine
series approximation to $f(x)$.
```

Example: Let

$$f(x) = \begin{cases} -1, & 0 \leq x \leq \pi/2, \\ 2, & \pi/2 < x < \pi. \end{cases}$$

Then $L = \pi$ and

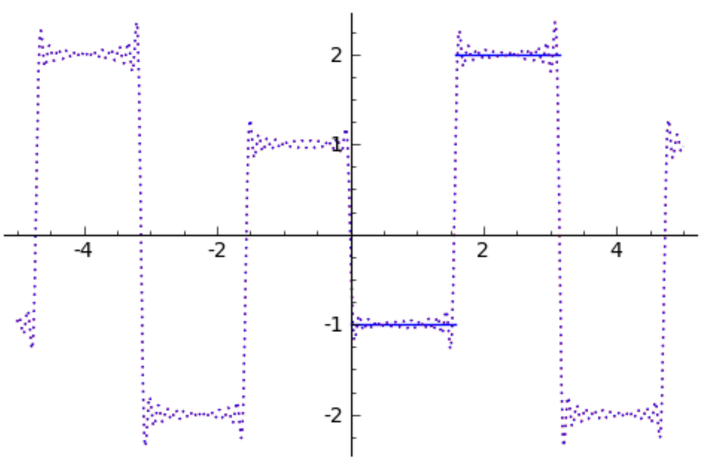
$$b_n(f) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx = -2\,\frac{2\,\cos(n\pi) - 3\,\cos(\frac{1}{2}\,n\pi) + 1}{n\pi}.$$

Thus

$$f(x) \sim b_1(f)\sin(x) + b_2(f)\sin(2x) + ... = \frac{2}{\pi}\sin(x) - \frac{6}{\pi}\sin(2x) + \frac{2}{3\pi}\sin(3x) + ...$$

This can also be done in Sage. First, we compute and plot several terms of the sine series approximation to $f(x)$.

```
x = var("x")
f1(x) = -1
f2(x) = 2
f = Piecewise([[(0,pi/2),f1],[(pi/2,pi),f2]])
P1 = f.plot()
b50 = [f.sine_series_coefficient(n,pi) for n in range(1,50)]
ss50 = sum([b50[n]*sin((n+1)*x) for n in range(len(b50))])
P2  = ss50.plot(-5,5,linestyle=":", rgbcolor=(1,0,0))
(P1+P2+P3).show()
```




```
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Now we use Sage to plot the solution $u(x,t)$ to the heat equation

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when $t=0$, $t=1/20$, $t=1/10$, $t=1/5$.
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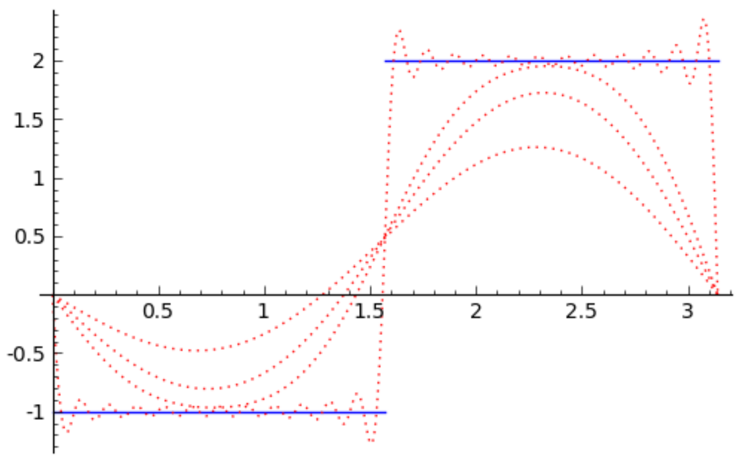
when $t=0$, $t=1/20$, $t=1/10$, $t=1/5$.

```
t = var("t")
soln50(t) = sum([b50[n]*sin((n+1)*x)*e^(-(n+1)^2*t) for n in range(49)])
P2  = ss50.plot(0, pi, linestyle=":", rgbcolor=(1,0,0))
P3 = plot(soln50(1/20) ,0,pi, linestyle=":", rgbcolor=(1,0,0))
P4 = plot(soln50(1/10)  ,0,pi, linestyle=":", rgbcolor=(1,0,0))
P5 = plot(soln50(1/5) ,0,pi, linestyle=":", rgbcolor=(1,0,0))
(P1+P2+P3+P4+P5).show()
```

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You can see the absolute value of the temperature on the wire decreasing with time, as
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Hope this helps show how Sage can be used to illustrate Fourier series and some of
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The End

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